Nonradial solutions for the Hénon equation in $\mathbb{R}^N$ *

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January 18, 2013

Abstract

In this paper we consider the problem

$$\begin{cases} -\Delta u = (N+\alpha)(N-2)|x|^\alpha u^{\frac{N+2+2\alpha}{N-2}} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in D^{1,2}\left(\mathbb{R}^N\right) \end{cases}$$

where $N \geq 3$. From the characterization of the solutions of the linearized operator, we deduce the existence of nonradial solutions which bifurcate from the radial one when α is an even integer.

Contents

*The first two authors are supported by PRIN-2009-WRJ3W7 grant, S. Neves has been partially supported by FAPESP.

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1 Introduction and statement of the main results

We consider the problem

$$\begin{cases} -\Delta u = C(\alpha)|x|^{\alpha}u^{p_\alpha} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in D^{1,2}\left(\mathbb{R}^N\right) \end{cases} \tag{1.1}$$

where $N \geq 3$, $\alpha > 0$, $p_\alpha = \frac{N+2+2\alpha}{N-2}$, $C(\alpha) = (N+\alpha)(N-2)$, $D^{1,2}\left(\mathbb{R}^N\right) = \{u \in L^{2^*}(\mathbb{R}^N)$ such that $|\nabla u| \in L^2(\mathbb{R}^N)\}$ and $2^* = \frac{2N}{N-2}$.
This problem, for $\alpha > 0$, generalizes the well-known equation which involves the critical Sobolev exponent

$$\begin{cases} -\Delta u = N(N-2)u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

Smooth solutions to (1.2) have been completely classified in [CGS89], where the authors proved that they are given by

$$U_{\lambda,y}(x) = \frac{\lambda^{\frac{N-2}{2}}}{\left(1+\lambda^2|x-y|^2\right)^{\frac{N-2}{2}}} \tag{1.3}$$

with $\lambda > 0$ and $y \in \mathbb{R}^N$ and they are extremal functions for the well-known Sobolev inequality,

$$\int_{\mathbb{R}^N} |\nabla u|^2 \geq S\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}. \tag{1.4}$$

The presence of the term $|x|^\alpha$ in equation (1.1) drastically changes the problem. For this kind of nonlinearities it is not possible to apply the moving plane method anymore (to get the radial symmetry around some point), and indeed nonradial solutions appear, as we will see in Theorem 1.6. This phenomenon has brought attention to the Hénon problem, i.e. (1.1) or

$$\begin{cases} -\Delta u = C(\alpha)|x|^{\alpha}u^{p} & \text{in } B_1 \\ u > 0 & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases} \tag{1.5}$$

where B_1 is the unit ball of $\mathbb{R}^N$, $N \geq 3$, and $p > 1$. Problem (1.5) was widely studied, mainly in the subcritical range $1 < p < \frac{N+2}{N-2}$ (see for example [SSW02], [PS07] and the references therein) where the existence of nonradial solutions was observed. The only result in the full range $(1, \frac{N+2+2\alpha}{N-2})$ is the following one (see W. M. Ni, [N82]) ,

Theorem 1.1. *Let B_1 be the unit ball of $\mathbb{R}^N$, $N \geq 3$. Then, for any $1 < p < \frac{N+2+2\alpha}{N-2}$ there exists a radial solution to the problem* (1.5)*. Moreover, if $p \geq \frac{N+2+2\alpha}{N-2}$ there exists no solution to* (1.5)*.*

Coming back to (1.1), we quote the following result by E. Lieb ([L83]), which, in the radial case, extends the inequality (1.4).

Theorem 1.2. *Let $u \in D^{1,2}\left(\mathbb{R}^N\right)$ be a radial function. Then we have that,*

$$\int_{\mathbb{R}^N} |\nabla u|^2 \geq S(\alpha)\left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{\frac{2N+2\alpha}{N-2}}\right)^{\frac{N-2}{N}}, \tag{1.6}$$

for some positive constant $S(\alpha)$. Moreover the extremal functions which achieve $S(\alpha)$ are solutions to (1.1) *and are given by*

$$U_{\lambda,\alpha}(x) = \frac{\lambda^{\frac{N-2}{2}}}{\left(1+\lambda^{2+\alpha}|x|^{2+\alpha}\right)^{\frac{N-2}{2+\alpha}}} \tag{1.7}$$

with $\lambda > 0$.

In ([GS81]) it was proved that the functions in (1.7) are the *unique* radial solutions to (1.1). We will call U_α the unique radial solution of (1.1), related to the exponent α, such that $U_\alpha(0) = 1$, i.e.

$$U_\alpha(x) = \frac{1}{\left(1+|x|^{2+\alpha}\right)^{\frac{N-2}{2+\alpha}}}. \tag{1.8}$$

In this paper we are interested in the existence of nonradial solutions for problem (1.1). This problem is quite difficult because, in this case, there is no embedding of the space $D^{1,2}\left(\mathbb{R}^N\right)$ in $L^{\frac{2N+2\alpha}{N-2}}\left(\mathbb{R}^N\right)$. So the standard variational methods can not be applied. To overcome this problem we will use the *bifurcation theory.* Our first result concerns the study of the linearized problem related to (1.1) at the function U_α. This leads to study the problem,

$$\begin{cases} -\Delta v = C(\alpha) p_\alpha |x|^\alpha U_\alpha^{p_\alpha - 1} v & \text{in } \mathbb{R}^N \\ v \in D^{1,2}\left(\mathbb{R}^N\right). \end{cases} \tag{1.9}$$

Next theorem characterizes all the solutions to (1.9).

Theorem 1.3. *Let $\alpha \geq 0$. If $\alpha > 0$ is not an even integer, then the space of solutions of* (1.9) *has dimension* 1 *and is spanned by*

$$Z(x) = \frac{1-|x|^{2+\alpha}}{\left(1+|x|^{2+\alpha}\right)^{\frac{N+\alpha}{2+\alpha}}}. \tag{1.10}$$

If $\alpha = 2(k-1)$ for some $k \in \mathbb{N}$ then the space of solutions of (1.9) *has dimension $1 + \frac{(N+2k-2)(N+k-3)!}{(N-2)!k!}$ and is spanned by the functions*

$$Z(x) = \frac{1-|x|^{2+\alpha}}{\left(1+|x|^{2+\alpha}\right)^{\frac{N+\alpha}{2+\alpha}}}\ ,\ Z_k(x) = \frac{Y_k(x)}{\left(1+|x|^{2+\alpha}\right)^{\frac{N+\alpha}{2+\alpha}}} \tag{1.11}$$

where Y_k form a basis of $\mathbb{Y}_k(\mathbb{R}^N)$, the space of all homogeneous harmonic polynomials of degree k in $\mathbb{R}^N$.

We note that in the case $\alpha = 0$ we get $k = 1$ and one gets back the known result for the equation involving the critical Sobolev exponent. We observe that for all $\alpha > 0$ the problem (1.1) is invariant for dilations but not for translations. Theorem 1.3 highlights the new phenomenon that if α is an even integer then there exist new solutions to (1.9) that "replace" the ones due to the translations invariance. It would be very interesting to understand if these new solutions are given by some geometrical invariants of the problem or not.
The key step of the proof is the change of variables $r \mapsto r^{\frac{2}{\alpha+2}}$. In this way the problem (1.9) leads back, in a suitable sense, to the well-known case $\alpha = 0$, where there is a complete characterization of the solutions.
We emphasize that the transformation $r \mapsto r^{\frac{2}{\alpha+2}}$, which was used in [CG10] in a different context, allows to prove in an easy way some known results.
The first example is a new (and in our opinion very simple) proof of the inequality (1.6), which also provides the uniqueness result due to Gidas and Spruck in [GS81]. The second one is a new proof of Theorem 1.1 jointly with the uniqueness of the radial solution (this last result was proved in [NN85]). Both proofs are given in the Appendix.
A first consequence of Theorem 1.3 is the computation of the Morse index of the solution U_α.

Corollary 1.4. *Let U_α be the solution of* (1.1)*, then its Morse index $m(\alpha)$ is equal to*

$$m(\alpha) = \sum_{\substack{0 \le k < \frac{\alpha+2}{2} \\ k \ integer}} \frac{(N+2k-2)(N+k-3)!}{(N-2)!\,k!}.$$

In particular, we have that the Morse index of U_α changes as α crosses the even integers and also that $m(\alpha) \to +\infty$ as $\alpha \to +\infty$.

Now let us consider the most important consequence of Theorem 1.3: the existence of *nonradial solutions* to (1.1).
Set $X = D^{1,2}(\mathbb{R}^N) \cap L^\infty_\beta(\mathbb{R}^N)$ where $L^\infty_\beta(\mathbb{R}^N)$ is a suitable $L^\infty-$weighted space (see (3.6), (3.8) for the precise definition). First let us give the following definition,

Definition 1.5. *Let U_α be the radial solution of* (1.1) *defined in* (1.8)*. We say that a nonradial bifurcation occurs at $(\overline{\alpha}, U_{\overline{\alpha}})$ if in every neighborhood of $(\overline{\alpha}, U_{\overline{\alpha}})$ in $(0, +\infty) \times X$ there exists a point (α, v_α) with v_α nonradial solution of* (1.1)*.*

Let $O(h)$ be the orthogonal group in $\mathbb{R}^h$. Our main result is the following.

Theorem 1.6. *Let $\alpha = 2(k-1)$ with $k \in \mathbb{N}$, $k \ge 2$. Then, (see Figure 1)*
i) If k is odd there exists at least a continuum of nonradial solutions to (1.1)*, invariant with respect to $O(N-1)$, bifurcating from the pair (α, U_α).*
ii) If k is even there exist at least $\left[\frac{N}{2}\right]$ continua of nonradial solutions to (1.1)*,*

invariant with respect to $O(N-1), O(N-2)\times O(2), \ldots\ldots,$ *bifurcating from the pair* (α, U_α).
Moreover all these solutions are fast decaying, in the sense that

$$\limsup_{|x|\to+\infty} |x|^{N-2} v(x) < +\infty.$$

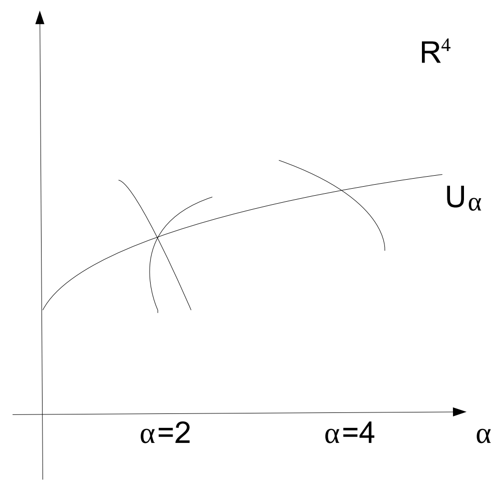

Figure 1

The previous theorem states that the structure of solutions to (1.1) is much more complex than the case $\alpha = 0$. In particular, it highlights the special role of the *even* numbers α. The proof of Theorem 1.6 requires a lot of work. In fact, even if Theorem 1.3 suggests the existence of nonradial bifurcation points, it is not possible to a pply directly the classical bifurcation theory, because the solution U_α is not isolated. This fact also makes very complicated to calculate the degree of the operator naturally associated with the problem, which is known as a crucial tool in the bifurcation theory.
In order to overcome these difficulties, we introduce a suitable approximated problem on balls of radius $\frac{1}{\epsilon}$ and there we apply the classical bifurcation theory. In this way we deduce the existence of nonradial solutions v_ϵ which bifurcate from some radial functions close to U_α. The final part of the proof will be to show that these solutions converge to nonradial solutions of problem (1.1) as $\epsilon \to 0$. This last part requires several delicate estimates. In particular, we emphasize that a careful use of the Pohozaev identity allow us to show that the approximated solutions v_ε stay away from the branches of radial solutions of the limit problem.
A natural question that arises from Theorem 1.6 is the study of the shape of the bifurcation diagram of the nonradial solutions. From the classical bifurcation theory we are not able to derive information of this type. However, we conjecture

that the nonradial solutions of Theorem 1.6 exist only when α is an even integer (see Figure 2).

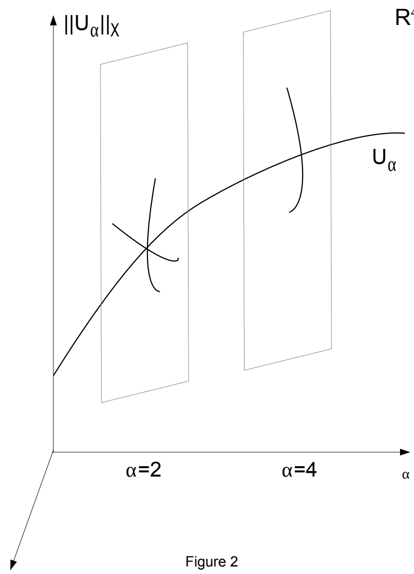

Figure 2

We have no proof of this, but we are going to make two remarks that support this conjecture. The first one is the calculation of a branch of explicit solutions which bifurcate from U_α, at least when $\alpha = 2$ and N is even.

Proposition 1.7. *Let $\alpha = 2$, $N \geq 4$ even and $x \in \mathbb{R}^N = \mathbb{R}^{\frac{N}{2}} \times \mathbb{R}^{\frac{N}{2}}$, $x = (x', x'')$ with $x' \in \mathbb{R}^{\frac{N}{2}}$ and $x'' \in \mathbb{R}^{\frac{N}{2}}$. Then, for any $a \in \mathbb{R}$, the functions*

$$u(x) = u(|x'|, |x''|) = \frac{1}{(1 + |x|^4 - 2a(|x'|^2 - |x''|^2) + a^2)^{\frac{N-2}{4}}} \tag{1.12}$$

form a branch of solutions to (1.1) *bifurcating from U_2.*

The second reason that supports our conjecture is the following classification result for a Liouville-type equations with singular data (see J. Prajapat and G. Tarantello [PT01]).

Theorem 1.8. *Let us consider the problem*

$$\begin{cases} -\Delta u = 2(\alpha+2)^2 |x|^\alpha e^u \quad \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |x|^\alpha e^u < +\infty. \end{cases} \tag{1.13}$$

If α is not an even *integer then the unique solutions to* (1.13) *are given by*

$$u_\lambda(x) = \log \frac{\lambda^{\alpha+2}}{(1+\lambda^{\alpha+2}|x|^{\alpha+2})^2}, \qquad \lambda > 0. \tag{1.14}$$

On the other hand, if α is an even *integer we also have the following nonradial solutions, for $a \in \mathbb{R}$, $\theta_0 \in [0, 2\pi]$*

$$v_{a,\theta_0}(x) = \log \frac{1}{\left(1+|x|^{\alpha+2} - 2a|x|^{\frac{\alpha+2}{2}} \cos\left(\frac{\alpha+2}{2}(\theta-\theta_0)\right) + a^2\right)^2}. \tag{1.15}$$

with θ the angle of x in polar coordinates.

Problem (1.13), that admits nonradial solutions only if α is an even integer, can be seen as the equivalent of (1.1) if $N = 2$.
Note also the similarities of (1.15), when $\alpha = 2$, with the explicit solution given by (1.12).
Another important similarity between our results and those related to problem (1.13), concerns the analogue of Theorem 1.3 ([DEM12]).

Theorem 1.9. *Let α be an even integer and $k = \frac{\alpha+2}{2}$. Let us consider the linearized problem associated to* (1.13)*, i.e.,*

$$-\Delta z = 2(\alpha+2)^2|x|^\alpha e^{u_1(x)} z \quad \text{in } \mathbb{R}^2 \tag{1.16}$$

Then the space of all bounded solutions of (1.16) *is spanned by*

$$Z_1(x) = \frac{1-|x|^{2+\alpha}}{1+|x|^{2+\alpha}}, \quad Z_2(x) = \frac{P_1(x)}{1+|x|^{2+\alpha}}, \quad Z_3(x) = \frac{P_2(x)}{1+|x|^{2+\alpha}} \tag{1.17}$$

where $P_1(x)$ and $P_2(x)$ form a basis of $\mathbb{Y}_k(\mathbb{R}^2)$, the space of all homogeneous harmonic polynomials of degree k in $\mathbb{R}^2$.

We conclude showing a further application of Theorem 1.3 that allows to give a generalization of an existence result of single-peak solutions for the almost critical Hénon equation (see [GG12]) in bounded domains.

Theorem 1.10. *Let $\alpha > 0$ different from an* even integer *and Ω be a smooth bounded domain of $\mathbb{R}^N$ with $N \geq 3$ and $0 \in \Omega$. Then, for ε small enough, there exists a solution u_ε to*

$$\begin{cases} -\Delta u = |x|^\alpha u^{\frac{N+2+2\alpha}{N-2}-\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

This theorem was proved in [GG12] under the more restrictive assumption $0 < \alpha \leq 1$, since it was not available the characterization of the solutions of (1.9). The proof is based on the Liapunov-Schmidt finite dimensional reduction

method and still works if α is different from an *even integer* since the kernel of the linearized operator is one-dimensional. The case of α even is much more difficult, due to the richness of the kernel of the linearized operator, and it seems difficult to handle as the previous one. Obviously, Theorem 1.3 can be applied to asymptotic problems similar to (1.18).

The paper is organized as follows. In Section 2 we prove Theorem 1.3 and Corollary 1.4. In Section 3 we study the approximated problem in the ball and we construct our approximated solution. In Section 4 we prove some estimates on the approximated solution which enable us to pass to the limit and in Section 5 we prove Theorem 1.6 and Proposition 1.7. Finally in the Appendix we give a simplified proof of known results.

2 The linearized operator

In this section we consider the linearized problem (1.8) and we give the proof of Theorem 1.3 and of Corollary 1.4.

Proof of Theorem 1.3. We want to find solutions of

$$\begin{cases} -\Delta V = C(\alpha) p_\alpha \frac{|x|^\alpha}{(1+|x|^{2+\alpha})^2} V & \text{in } \mathbb{R}^N \\ V \in D^{1,2}\left(\mathbb{R}^N\right) \end{cases} \tag{2.1}$$

of the form

$$V(r,\theta) = \sum_{k=0}^{\infty} \psi_k(r) Y_k(\theta), \qquad \text{where} \qquad r = |x| \ , \ \theta = \frac{x}{|x|} \in S^{N-1}$$

and

$$\psi_k(r) = \int_{S^{N-1}} V(r,\theta) Y_k(\theta) d\theta.$$

Here $Y_k(\theta)$ denotes the k-th spherical harmonics, i.e. it satisfies

$$-\Delta_{S^{N-1}} Y_k = \mu_k Y_k$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1} with the standard metric and μ_k is the k-th eigenvalue of $-\Delta_{S^{N-1}}$. It is known that

$$\mu_k = k(N-2+k), \qquad k = 0, 1, 2, \ldots$$

whose multiplicity is

$$\frac{(N+2k-2)(N+k-3)!}{(N-2)!\, k!}$$

and that

$$Ker\left(\Delta_{S^{N-1}} + \mu_k\right) = \mathbb{Y}_k(\mathbb{R}^N)\,|_{S^{N-1}}\ .$$

The function V is a solution of (1.9) if and only if $\psi_k(r)$ satisfies

$$\begin{cases} -\psi_k''(r) - \frac{N-1}{r}\psi_k'(r) + \frac{\mu_k}{r^2}\psi_k(r) = C(\alpha)p_\alpha \frac{r^\alpha}{(1+r^{2+\alpha})^2}\,\psi_k(r)\,, \quad \text{in } (0,\infty) \\ \psi_k \in \mathcal{E} \\ \psi_k'(0)=0 \text{ if } k=0 \quad \text{and} \quad \psi_k(0)=0 \text{ if } k\geq 1 \end{cases} \tag{2.2}$$

where

$$\mathcal{E} = \left\{ \psi \in C^1([0,\infty)) \;\;\middle|\;\; \int_0^\infty r^{N-1}|\psi'(r)|^2 dr < \infty \right\}. \tag{2.3}$$

We shall solve (2.2) using the following change of variables

$$\eta_k(r) = \psi_k(r^{\frac{2}{2+\alpha}}) \tag{2.4}$$

that transforms (2.2) into the equation

$$\begin{cases} -\eta_k''(r) - \frac{M-1}{r}\eta_k'(r) + \frac{4\mu_k}{(2+\alpha)^2}\frac{\eta_k(r)}{r^2} = M(M+2)\,\frac{\eta_k(r)}{(1+r^2)^2}\,, \quad \text{in } (0,\infty) \\ \eta_k \in \widetilde{\mathcal{E}} \\ \eta_k'(0)=0 \text{ if } k=0 \quad \text{and} \quad \eta_k(0)=0 \text{ if } k\geq 1 \end{cases} \tag{2.5}$$

where

$$M = \frac{2(N+\alpha)}{2+\alpha}\,, \qquad \widetilde{\mathcal{E}} = \left\{ \eta \in C^1([0,\infty)) \;\;\middle|\;\; \int_0^\infty r^{M-1}|\eta'(r)|^2 dr < \infty \right\}. \tag{2.6}$$

Note that we have $\eta_0'(0)=0$ since $\eta_0'(r) = \frac{2}{\alpha+2}r^{-\frac{\alpha}{\alpha+2}}\psi_0'\left(r^{\frac{2}{\alpha+2}}\right)$ and by (2.2) $\psi_0'(r) = O(r^{\alpha+1})$ near $r=0$.

Fixed M let us now consider the following eigenvalue problem

$$-\eta''(r) - \frac{M-1}{r}\eta'(r) + \beta\frac{\eta(r)}{r^2} = M(M+2)\,\frac{\eta(r)}{(1+r^2)^2}\,, \quad \text{in } (0,\infty). \tag{2.7}$$

The equation (2.7) is a singular Sturm-Liouville problem, it has a sequence of simple eigenvalues $\beta_1 > \beta_2 > \dots$ and if η_j is an eigenfunction of β_j then η_j has exactly $j-1$ zeros in $(0,\infty)$, see for example [Z05, Theorem 10.12.1].

When M is an integer we can study (2.7) as the linearized operator of the equation $-\Delta U = U^{\frac{M+2}{M-2}}$ around the standard solution $U(x) = \frac{1}{(1+|x|^2)^{\frac{M-2}{2}}}$, (note that we always have $M>2$). In this case, we know that

$$\beta_1 = M-1\,;\; \beta_2 = 0 \quad \text{and} \quad \eta_1(r) = \frac{r}{(1+r^2)^{\frac{M}{2}}}\;;\; \eta_2(r) = \frac{1-r^2}{(1+r^2)^{\frac{M}{2}}}\,. \tag{2.8}$$

However, even when M is not an integer we readily see that (2.8) remains true. Therefore, we can conclude that (2.5) has nontrivial solutions if and only if

$$\frac{4\mu_k}{(2+\alpha)^2} \in \{0\,, M-1\},$$

which means that

$$k = 0 \quad \text{or} \quad \alpha = 2(k-1).$$

Turning back to (2.2) we obtain the solutions

$$\psi_0(r) = \frac{1 - r^{2+\alpha}}{\left(1 + r^{2+\alpha}\right)^{\frac{N+\alpha}{2+\alpha}}} \qquad \text{if } \alpha \neq 2(k-1)\ ,\ k \in \mathbb{N} \tag{2.9}$$

$$\psi_0(r) = \frac{1 - r^{2+\alpha}}{\left(1 + r^{2+\alpha}\right)^{\frac{N+\alpha}{2+\alpha}}}\ ,\ \psi_k(r) = \frac{r^k}{\left(1 + r^{2+\alpha}\right)^{\frac{N+\alpha}{2+\alpha}}} \qquad \text{if } \alpha = 2(k-1)\ ,\ k \in \mathbb{N} \tag{2.10}$$

and the proof of the theorem is complete. □

Proof of Corollary 1.4. Let us now consider the following eigenvalue problem,

$$\begin{cases} -\Delta V = \Lambda C(\alpha) p_\alpha \frac{|x|^\alpha}{(1+|x|^{2+\alpha})^2} V & \text{in } \mathbb{R}^N \\ V \in D^{1,2}\left(\mathbb{R}^N\right). \end{cases} \tag{2.11}$$

The Morse index of U_α is the sum of the dimensions of the eigenspaces of (2.11) related to $\Lambda < 1$.

As in the proof of Theorem 1.3, we are led to the equation

$$\begin{cases} -\psi_k''(r) - \frac{N-1}{r}\psi_k'(r) + \frac{\mu_k}{r^2}\psi_k(r) = \Lambda C(\alpha) p_\alpha \frac{r^\alpha}{(1+r^{2+\alpha})^2}\, \psi_k(r)\,, & \text{in } (0,\infty) \\ \psi_k \in \mathcal{E}, \quad \Lambda < 1 \\ \psi_k'(0) = 0 \ \text{if } k = 0 \quad \text{and} \quad \psi_k(0) = 0 \ \text{if } k \geq 1. \end{cases} \tag{2.12}$$

For every $k \geq 0$, the problem (2.12) has an increasing sequence of eigenvalues $\Lambda_{j,k}$, $j = 1, 2, \ldots$ and an associated eigenfunction has exactly $j-1$ zeros in $(0,\infty)$.

We claim that $\Lambda_{j,k} \geq 1$ for all $j \geq 2$ and any $k \geq 0$. Indeed, using the transformation (2.4), as in the previous theorem we get

$$\begin{cases} -\eta_k''(r) - \frac{M-1}{r}\eta_k'(r) + \frac{4\mu_k}{(2+\alpha)^2}\,\frac{\eta_k(r)}{r^2} = \Lambda M(M+2)\,\frac{\eta_k(r)}{(1+r^2)^2}\,, & \text{in } (0,\infty) \\ \eta_k \in \widetilde{\mathcal{E}} \\ \eta_k'(0) = 0 \ \text{if } k = 0 \quad \text{and} \quad \eta_k(0) = 0 \ \text{if } k \geq 1 \end{cases} \tag{2.13}$$

and, for any fixed $\Lambda < 1$, we consider the more general equation

$$-\eta''(r) - \frac{M-1}{r}\eta'(r) + \beta\frac{\eta(r)}{r^2} = \Lambda M(M+2)\,\frac{\eta(r)}{(1+r^2)^2}\,, \quad \text{in } (0,\infty) \tag{2.14}$$

which has a sequence of eigenvalues $\{\beta_1(\Lambda) > \beta_2(\Lambda) > \ldots\}$. Since $\Lambda < 1$, using the min-max characterization of the eigenvalues and (2.8) we infer that

$$M - 1 = \beta_1(1) > \beta_1(\Lambda) \qquad \text{and} \qquad 0 = \beta_2(1) > \beta_2(\Lambda). \tag{2.15}$$

Now, let $\psi_{j,k}$ be an eigenfunction of (2.12) related to $\Lambda_{j,k}$ for some $j \geq 2$, $k \geq 0$ and suppose that $\Lambda_{j,k} < 1$. Then $\psi_{j,k}(r^{\frac{2}{2+\alpha}})$ is an eigenfunction of (2.13) which changes sign related to $\beta(\Lambda_{j,k}) = \frac{4\mu_k}{(2+\alpha)^2} \geq 0$. On the other hand, by (2.15) with $\Lambda = \Lambda_{j,k} < 1$ we have that $\beta(\Lambda_{j,k}) \leq \beta_2(\Lambda) < 0$, and a contradiction arises. Hence the claim is proved.

By the previous claim it follows that $j = 1$ and therefore only the first eigenvalues of (2.12) may contribute to the Morse index. When α is an even integer, from (2.10), we already know that

$$\psi_k(r) = \frac{r^k}{(1+r^{2+\alpha})^{\frac{N+\alpha}{2+\alpha}}} = \frac{r^k}{(1+r^{2+\alpha})^{\frac{N+2(k-1)}{2+\alpha}}}$$

is the first eigenfunction of (2.12), for $k = \frac{\alpha+2}{2}$, with eigenvalue $\Lambda = 1$. However, for general $\alpha > 0$ and $k \geq 0$, we can check that the first eigenfunction of (2.12) is still

$$\psi_{1,k}(r) = \frac{r^k}{(1+r^{2+\alpha})^{\frac{N+2(k-1)}{2+\alpha}}} \tag{2.16}$$

and the first eigenvalue of (2.12) is

$$\Lambda_{1,k} = \frac{(N-2+2k)(N+\alpha+2k)}{(N+2+2\alpha)(N+\alpha)}. \tag{2.17}$$

A straightforward computation shows that

$$\Lambda_{1,k} < 1 \quad \Leftrightarrow \quad k < \frac{\alpha+2}{2}.$$

Therefore, the eigenvalues of (2.11) that satisfy $\Lambda < 1$ are precisely $\Lambda_{1,k}$ for $k < \frac{\alpha+2}{2}$ and the eigenfunctions of (2.11) related to $\Lambda_{1,k}$ are linear combinations of functions of the form

$$V = \frac{|x|^k}{(1+|x|^{2+\alpha})^{\frac{N+2(k-1)}{2+\alpha}}} Y_k\left(\frac{x}{|x|}\right) \qquad \text{where} \quad Y_k \in \mathbb{Y}_k(\mathbb{R}^N). \tag{2.18}$$

Since $\dim(\mathbb{Y}_k(\mathbb{R}^N)) = \frac{(N+2k-2)(N+k-3)!}{(N-2)!\,k!}$ the proof is now complete. $\square$

3 The approximated problem

In this section we consider the problem

$$\begin{cases} -\Delta u = C(\alpha)|x|^\alpha \left(u + U_\alpha(\frac{1}{\varepsilon})\right)^{p_\alpha}, & \text{in } B_{\frac{1}{\varepsilon}}(0), \\ u \geq 0 & \text{in } B_{\frac{1}{\varepsilon}}(0), \\ u = 0, & \text{on } \partial B_{\frac{1}{\varepsilon}}(0), \end{cases} \tag{3.1}$$

where $\alpha > 0$ is fixed and $B_{\frac{1}{\varepsilon}}(0)$ denotes the ball of radius $\frac{1}{\varepsilon}$ centered at the origin. We let

$$u_{\varepsilon,\alpha}(x) = U_\alpha(x) - U_\alpha\left(\frac{1}{\varepsilon}\right) = \frac{1}{(1+|x|^{2+\alpha})^{\frac{N-2}{2+\alpha}}} - \frac{\varepsilon^{N-2}}{(1+\varepsilon^{2+\alpha})^{\frac{N-2}{2+\alpha}}} \tag{3.2}$$

where $U_\alpha(x)$ is as defined in (1.8). We have

Lemma 3.1. *For any $\alpha \geq 0$ and for any $0 < \varepsilon < 1$ sufficiently small, the function $u_{\varepsilon,\alpha}$ is a radial solution of* (3.1) *which is nondegenerate in the space of the radial functions.*

Proof. For any $\varepsilon > 0$ and for any $x \in B_{\frac{1}{\varepsilon}}(0)$ it is immediate to check that $u_{\varepsilon,\alpha}$ is a radial solution to (3.1). The function $u_{\varepsilon,\alpha}$ is radially nondegenerate if the linearized problem

$$\begin{cases} -\Delta\varphi = p_\alpha C(\alpha)|x|^\alpha \left(u_{\varepsilon,\alpha} + U_\alpha(\frac{1}{\varepsilon})\right)^{p_\alpha - 1}\varphi, & \text{in } B_{\frac{1}{\varepsilon}}(0), \\ \varphi = 0, & \text{on } \partial B_{\frac{1}{\varepsilon}}(0), \end{cases} \tag{3.3}$$

does not have radial nontrivial solutions. Using (3.2) we can rewrite (3.3) in radial coordinates, i.e.

$$\begin{cases} -(r^{N-1}\varphi')' = p_\alpha C(\alpha)\frac{r^{\alpha+N-1}}{(1+r^{2+\alpha})^2}\varphi, & \text{in } (0, \frac{1}{\varepsilon}), \\ \varphi'(0) = 0, \quad \varphi(\frac{1}{\varepsilon}) = 0. \end{cases} \tag{3.4}$$

Observe that the function $z(r) := \frac{1-r^{2+\alpha}}{(1+r^{2+\alpha})^{\frac{N+\alpha}{2+\alpha}}}$ satisfies the linearized equation

$$-(r^{N-1}z')' = p_\alpha C(\alpha)\frac{r^{\alpha+N-1}}{\left(1+r^{2+\alpha}\right)^2} z, \text{ in } \mathbb{R}_+, \qquad z'(0) = 0 \tag{3.5}$$

but $z(\frac{1}{\varepsilon}) \neq 0$, for $\varepsilon < 1$.

Multiplying (3.4) by z, (3.5) by φ and integrating we get

$$\varphi'\left(\frac{1}{\varepsilon}\right) = 0$$

and since $\varphi(\frac{1}{\varepsilon}) = 0$ we must have $\varphi \equiv 0$. This implies that (3.4) does not have any nontrivial solution and hence $u_{\varepsilon,\alpha}$ is radially nondegenerate. □

Let $\beta > 0$ be such that $\frac{N}{N+2}(N-2) < \beta < N-2$. For every $g \in L^\infty(\mathbb{R}^N)$, we define the weighted norm

$$\|g\|_\beta := \sup_{x\in\mathbb{R}^N}(1+|x|)^\beta|g(x)|. \tag{3.6}$$

Set $L^\infty_\beta(\mathbb{R}^N) := \{g \in L^\infty(\mathbb{R}^N) \text{ such that } \exists\, C > 0 \text{ and } \|g\|_\beta < C\}$ and

$$X = D^{1,2}(\mathbb{R}^N) \cap L^\infty_\beta(\mathbb{R}^N). \tag{3.7}$$

X is a Banach space with the norm

$$\|g\|_X := \max\{\|g\|_{1,2}, \|g\|_\beta\} \tag{3.8}$$

where $\|\cdot\|_{1,2}$ denotes the usual norm in $D^{1,2}(\mathbb{R}^N)$, i.e. $\|g\|_{1,2} = \left(\int_{\mathbb{R}^N}|\nabla g|^2\,dx\right)^{\frac{1}{2}}$ for $g \in D^{1,2}(\mathbb{R}^N)$. Now we are in position to state the following

Proposition 3.2. *Let ε_n and α_n be sequences such that $\varepsilon_n \to 0$ and $\alpha_n \to \alpha > 0$ as $n \to +\infty$. Denoting by $u_n := u_{\varepsilon_n,\alpha_n}$, with $u_{\varepsilon,\alpha}$ as defined in (3.2), we have that*

$$\|u_n - U_\alpha\|_X \to 0 \qquad \text{as } n \to +\infty \tag{3.9}$$

where u_n is assumed to be extended by zero outside $B_{\frac{1}{\varepsilon_n}}$.

Proof. By (3.8) we need to prove that $\|u_n - U_\alpha\|_{1,2} \to 0$ and $\|u_n - U_\alpha\|_\beta \to 0$ as $n \to +\infty$. By the definition of u_n, letting $B_n := B_{\frac{1}{\varepsilon_n}}$, we have

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla U_\alpha|^2\,dx = \int_{B_n} |\nabla U_{\alpha_n} - \nabla U_\alpha|^2\,dx + \int_{\mathbb{R}^N\setminus B_n} |\nabla U_\alpha|^2\,dx.$$

Since $U_\alpha \in D^{1,2}(\mathbb{R}^N)$ it follows that

$$\int_{\mathbb{R}^N\setminus B_n} |\nabla U_\alpha|^2\,dx \to 0 \qquad \text{as } n \to +\infty$$

and by the decay of $|\nabla U_\alpha|$

$$\int_{B_n} |\nabla U_{\alpha_n} - \nabla U_\alpha|^2\,dx \to 0 \qquad \text{as } n \to +\infty.$$

Finally, using the definitions of u_n, U_α and the mean value Theorem we have

$$\begin{aligned}
\|u_n - U_\alpha\|_\beta &= \sup_{x\in\mathbb{R}^N} (1+|x|)^\beta |u_n(x) - U_\alpha(x)| \\
&\le \sup_{x\in B_n} (1+|x|)^\beta\, |U_{\alpha_n}(x) - U_\alpha(x)| + \sup_{x\in B_n} (1+|x|)^\beta U_{\alpha_n}\left(\frac{1}{\varepsilon_n}\right) \\
&\quad + \sup_{x\in\mathbb{R}^N\setminus B_n} \frac{(1+|x|)^\beta}{(1+|x|^{2+\alpha})^{\frac{N-2}{2+\alpha}}} \\
&\le O(|\alpha-\alpha_n|) + O(\varepsilon_n^{N-2-\beta})
\end{aligned}$$

and since $\beta < N-2$, the proposition follows. ∎

3.1 Convergence of the spectrum

Let us consider the eigenvalue problem associated with (3.1), i.e.

$$\begin{cases} -\Delta z = \Lambda p_\alpha C(\alpha)|x|^\alpha \left(u_{\varepsilon,\alpha} + U_\alpha(\frac{1}{\varepsilon})\right)^{p_\alpha-1} z, & \text{in } B_{\frac{1}{\varepsilon}}(0), \\ z = 0, & \text{on } \partial B_{\frac{1}{\varepsilon}}(0). \end{cases} \tag{3.10}$$

Recalling that $u_{\varepsilon,\alpha} = U_\alpha - U_\alpha(\frac{1}{\varepsilon})$ we can rewrite (3.10) in the following way

$$\begin{cases} -\Delta z = \Lambda p_\alpha C(\alpha)\frac{|x|^\alpha}{(1+|x|^{2+\alpha})^2} z, & \text{in } B_{\frac{1}{\varepsilon}}(0), \\ z = 0, & \text{on } \partial B_{\frac{1}{\varepsilon}}(0). \end{cases} \tag{3.11}$$

We can decompose problem (3.11) in radial part and angular part using the spherical harmonic functions (as in Section 2), getting that z is a solution of (3.11) if and only if $z_k(r) := \int_{S^{N-1}} z(r,\theta)Y_k(\theta)\,d\theta$ is a solution of

$$\begin{cases} -z_k'' - \frac{N-1}{r}z_k' + \frac{\mu_k}{r^2}z_k = \Lambda p_\alpha C(\alpha)\frac{r^\alpha}{(1+r^{2+\alpha})^2}z, & \text{in } (0,\frac{1}{\varepsilon}), \\ z_k'(0) = 0 = z_k(\frac{1}{\varepsilon}) \text{ if } k=0 \quad \text{and} \quad z_k(0) = 0 = z_k(\frac{1}{\varepsilon}) \text{ if } k \geq 1 \end{cases} \tag{3.12}$$

for some $\mu_k = k(N+k-2)$, where $Y_k(\theta)$ denotes the k-th spherical harmonic function. In this way we have that all the eigenfunctions of (3.10) are given by $z_k(r)Y_k(\theta)$ if z_k is a solution of (3.12) related to some Λ.
Problem (3.12) is a singular Sturm-Liouville eigenvalue problem, for any $k \geq 0$, it has a sequence of eigenvalues $\Lambda_{i,k}^\varepsilon(\alpha)$, $i \in \mathbb{N}$ which are simple.
We have the following

Lemma 3.3. *Let ε_n be a sequence such that $\varepsilon_n \to 0$ as $n \to +\infty$. Let $\Lambda_{i,k}^n(\alpha)$ denote the i-th eigenvalue of (3.12) in $(0,\frac{1}{\varepsilon_n})$ (related to the exponent α), corresponding to μ_k, for some $k \geq 0$. Then*

$$\Lambda_{i,k}^n(\alpha) \to \Lambda_{i,k}(\alpha) \qquad \text{as } n \to +\infty \tag{3.13}$$

where $\Lambda_{i,k}(\alpha)$ is the i-th eigenvalue of the problem

$$\begin{cases} -z_k'' - \frac{N-1}{r}z_k' + \frac{\mu_k}{r^2}z_k = \Lambda p_\alpha C(\alpha)\frac{r^\alpha}{(1+r^{2+\alpha})^2}z_k, & \text{in } (0,+\infty), \\ z_k \in \mathcal{E} \\ z_k'(0) = 0 \text{ if } k=0 \quad \text{and} \quad z_k(0) = 0 \text{ if } k \geq 1 \end{cases} \tag{3.14}$$

and $\mathcal{E}$ as in (2.3)*. Moreover alle the eigenvalues $\Lambda_{i,k}^n(\alpha)$ and $\Lambda_{i,k}(\alpha)$ are analytic in α and the convergence in* (3.13) *is uniform in α on compact sets of* $(0,+\infty)$.

Proof. The convergence in (3.13) follows from the dependence on the domain of the eigenvalues or from the Sturm-Liouville theory (see [BEWZ93, Theorems 5.3 and 6.4] for example).
From a result of [K95, Theorem VII.3.9 or example in p.380] it follows that for any $n \in \mathbb{N}$ the eigenvalues $\Lambda_{i,k}^n(\alpha)$ and $\Lambda_{i,k}(\alpha)$ of the problem (3.12) and (3.14) are analytic in α.
For sake of simplicity, we will prove the uniform convergence in (3.13) only in the case $k=0$ and $i=2$, since it is what we need in the sequel. The case $k>0$ and $i \neq 2$ can be handled in a similar way.
So let $z_2^n(r)$ be the second eigenfunction of (3.12) corresponding to $k=0$, related to $\Lambda_{2,0}^n(\alpha)$ and normalized with the L^∞-norm. Integrating (3.12) on $(0,r)$ we get

$$-\left(r^{N-1}\left(z_2^n(r)\right)'\right) = \Lambda_{2,0}^n(\alpha)p_\alpha C(\alpha)\int_0^r \frac{s^{N-1+\alpha}}{\left(1+s^{2+\alpha}\right)^2}z_2^n(s)\,ds. \tag{3.15}$$

Observe that $\Lambda_{2,0}^n(\alpha) \leq \Lambda_{2,0}(\alpha)$ and that $\Lambda_{2,0}(\alpha)$ is uniformly bounded on com-

pact sets of $(0,+\infty)$ since it is analytic in α. Since $|z_2^n(r)|\le 1$ we get

$$| \left(z_2^n(r)\right)' | \le \frac{C}{r^{N-1}} \begin{cases} C & \text{if } N<4+\alpha \\ C+\log r & \text{if } N=4+\alpha \\ C+r^{N-4-\alpha} & \text{if } N>4+\alpha \end{cases} \tag{3.16}$$

If $N<4+\alpha$ we get the optimal decay for $z_2^n(r)$ and $\left(z_2^n(r)\right)'$, i.e.

$$| \left(z_2^n(r)\right)' | \le Cr^{1-N} \qquad |z_2^n(r)| \le Cr^{2-N} \tag{3.17}$$

if r is large enough. If else $N\ge 4+\alpha$, inserting (3.16) into (3.15) and iterating the previous procedure, after a finite number of steps we get (3.17) for any n and for any α on compact sets of $(0,+\infty)$. From (3.12) and (3.17) we have that

$$\int_0^{+\infty} r^{N-1} | \left(z_2^n(r)\right)' |^2 \, dr \le C$$

(where z_2^n is assumed to be zero for $r>\frac{1}{\varepsilon_n}$) so that $z_2^n \to \bar z_2$ in $\mathcal{E}$ (weakly), a.e. in $(0,+\infty)$ and uniformly on compact sets of $[0,+\infty)$. Using (3.17) again, we can pass to the limit into (3.12) getting that $\bar z_2$ is a solution of (3.14) corresponding to the eigenvalue $\Lambda_{2,0}(\alpha)$. Moreover $\bar z_2\ne 0$ since, from (3.17), the maximum point of $|z_2^n(r)|$ converges to a point $r_0\in[0,+\infty)$ and $|\bar z_2(r_0)|=1$ from the uniform convergence.
Finally, we multiply (3.12) by $\bar z_2$ and we integrate on $(0,\frac{1}{\varepsilon_n})$, we multiply (3.14) by z_2^n and we integrate on $(0,\frac{1}{\varepsilon_n})$, then we subtract getting

$$\begin{aligned} & -\left(\frac{1}{\varepsilon_n}\right)^{N-1} (z_2^n)'\left(\frac{1}{\varepsilon_n}\right) \bar z_2\left(\frac{1}{\varepsilon_n}\right) \\ & = \left[\Lambda_{2,0}^n(\alpha)-\Lambda_{2,0}(\alpha)\right] p_\alpha C(\alpha) \int_0^{\frac{1}{\varepsilon_n}} \frac{r^{N-1+\alpha}}{\left(1+r^{2+\alpha}\right)^2} z_2^n(r)\bar z_2(r)\, dr. \end{aligned}$$

Then

$$\left(\frac{1}{\varepsilon_n}\right)^{N-1} (z_2^n)'\left(\frac{1}{\varepsilon_n}\right) \bar z_2\left(\frac{1}{\varepsilon_n}\right) = O\left(\varepsilon_n^{N-2}\right)$$

as $n\to+\infty$, uniformly in α on compact sets of $(0,+\infty)$, while

$$p_\alpha C(\alpha) \int_0^{\frac{1}{\varepsilon_n}} \frac{r^{N-1+\alpha}}{\left(1+r^{2+\alpha}\right)^2} z_2^n(r)\bar z_2(r)\, dr \to \int_0^{+\infty} \frac{r^{N-1+\alpha}}{\left(1+r^{2+\alpha}\right)^2} \left(\bar z_2(r)\right)^2 dr > 0$$

as $n\to+\infty$, uniformly in α on compact sets of $(0,+\infty)$. This implies that

$$\sup_{\alpha\in K} \left|\Lambda_{2,0}^n(\alpha)-\Lambda_{2,0}(\alpha)\right| = o(1)$$

as $n\to+\infty$, for any compact set $K\subset(0,+\infty)$ and this concludes the proof.

□

Remark 3.4. Denoting by $\alpha_k = 2(k-1)$, for $k \geq 2$, and by $\Lambda_{i,k}(\alpha)$ the i-th eigenvalue of the problem (3.14), Theorem 1.3 implies that $\Lambda_{1,k}(\alpha_k) = 1$. From (2.17) we get that $\Lambda_{1,k}(\alpha_k - \delta) > 1$, $\Lambda_{1,k}(\alpha_k + \delta) < 1$ for $\delta > 0$ small enough, and from the proof of Corollary 1.4, $\Lambda_{2,k}(\alpha) > 1$ for any $k \geq 1$ and any $\alpha > 0$. This implies that the Morse index of U_α changes as α crosses α_k (see Corollary 1.4). In particular, passing from $\alpha_k - \delta$ to $\alpha_k + \delta$, for δ small enough, the Morse index of U_α increases of the dimension of the eigenspace $Ker(\Delta_{S^{N-1}} + \mu_k)$.
Using Lemma 3.3 we can deduce that

$$\Lambda^n_{1,k}(\alpha_k - \delta) > 1 \quad , \quad \Lambda^n_{1,k}(\alpha_k + \delta) < 1 \quad \text{and} \quad \Lambda^n_{2,k}(\alpha) > 1 \text{ for } \alpha \in [\alpha_k - \delta, \alpha_k + \delta]$$

if n is large enough. Then there exist points $\alpha^n_k \in [\alpha_k - \delta, \alpha_k + \delta]$ such that $\Lambda^n_{1,k}(\alpha^n_k) = 1$ and the function $\Lambda^n_{1,k}(\alpha) - 1$ changes sign in $\alpha = \alpha^n_k$. We can infer that the Morse index of the radial solution $u_{n,\alpha}$ changes in $\alpha = \alpha^n_k$. This implies that the Morse index of $u_{n,\alpha}$ increases (or decreases), passing from $\alpha^n_k - \delta'$ to $\alpha^n_k + \delta'$ and δ' small enough, exactly of the dimension of the eigenspace $Ker(\Delta_{S^{N-1}} + \mu_k)$. Moreover the analyticity of the eigenvalues in Lemma 3.3 implies that α^n_k are isolated.
Proceeding as in [BCGP12] it should be possible to show that $\Lambda^n_{1,k}(\alpha)$ is strictly decreasing in α (as $\Lambda_{1,k}(\alpha)$). In that case we can deduce that $\Lambda^n_{1,k}(\alpha) - 1$ changes sign at one point in the interval $[\alpha_k - \delta, \alpha_k + \delta]$.
Nevertheless, for our purposes, it is sufficient to use the analyticity of $\Lambda^n_{1,k}(\alpha)$ to say that in the interval $[\alpha_k - \delta, \alpha_k + \delta]$ the function $\Lambda^n_{1,k}(\alpha) - 1$ changes sign finitely many times, in particular in an odd number of exponents.
We will call α^n_k *Morse index changing exponents* for $u_{n,\alpha}$ and *Morse index changing points* the pairs $(\alpha^n_k, u_{n,\alpha^n_k})$, where u_{n,α^n_k} is the radial solution of (3.1) in B_n corresponding to the exponent α^k_n as defined in (3.2).

3.2 The bifurcation result in the ball

To state the bifurcation result we need some notations. As before we denote by $u_{n,\alpha}$ the radial solution of (3.1) corresponding to the exponent α, for $\varepsilon = \varepsilon_n$, and by B_n the ball of radius $\frac{1}{\varepsilon_n}$ centered at the origin. We will denote by

$$\mathcal{S}(n) := \left\{ \begin{array}{c} (\alpha, u_{n,\alpha}) \in (0, +\infty) \times C^{1,\gamma}_0(\overline{B}_n) \text{ such that } u_{n,\alpha} \\ \text{is the radial solution of (3.1) defined in (3.2)} \end{array} \right\} \tag{3.18}$$

Let us recall that, given the curve $\mathcal{S}(n)$, a point $(\alpha_i, u_{n,\alpha_i}) \in \mathcal{S}(n)$ is a *nonradial bifurcation point* if in every neighborhood of $(\alpha_i, u_{n,\alpha_i})$ in $(0, +\infty) \times C^{1,\gamma}_0(\overline{B}_n)$ there exists a point $(\alpha, v_{n,\alpha})$ such that $v_{n,\alpha}$ is a nonradial solution of (3.1) in B_n.
As in the previous Section (see Remark 3.4) we will call *Morse index changing points* the pair $(\alpha^n_k, u_{n,\alpha^n_k}) \in \mathcal{S}(n)$ in which the Morse index of the radial solution $u_{n,\alpha}$ changes.
We are in position to state our first result.

Theorem 3.5. *Let us fix $n \in \mathbb{N}$. Then the Morse index changing points are nonradial bifurcation points for the curve $\mathcal{S}(n)$.*

Proof. Let $(\alpha_k^n, u_{n,\alpha_k^n})$ be a Morse index changing point. We restrict our attention to the subspace $\mathcal{H}_n$ of $C_0^{1,\gamma}(\overline{B}_n)$ given by

$$\mathcal{H}_n := \{v \in C_0^{1,\gamma}(\overline{B}_n)\,,\ \text{s.t. } v(x_1,\dots,x_N) = v(g(x_1,\dots,x_{N-1}),x_N)\,,\ \text{for any } g \in O(N-1)\} \tag{3.19}$$

where $O(N-1)$ is the orthogonal group in $\mathbb{R}^{N-1}$.
By a result of Smoller and Wasserman in [SW86], we have that for any k the eigenspace V_k of the Laplace-Beltrami operator on S^{N-1}, spanned by the eigenfunctions corresponding to the eigenvalue μ_k which are $O(N-1)$ invariant, is one-dimensional. This implies that, (see Section 3.1), by the analyticity of $\Lambda_{1,k}^n(\alpha)$,

$$|m(\alpha_k^n+\delta) - m(\alpha_k^n-\delta)| = 1 \tag{3.20}$$

if $\delta > 0$ is small enough, where $m(\alpha)$ is the Morse index of the radial solution $u_{n,\alpha}$ in the space $\mathcal{H}_n$.
Let us consider the operator $T^n(\alpha,v) : (0,+\infty)\times\mathcal{H}_n \to \mathcal{H}_n$, defined by $T^n(\alpha,v) := (-\Delta)^{-1}\left(|x|^\alpha|v+\gamma_\alpha(n)|^{p_\alpha-1}(v+\gamma_\alpha(n))\right)$, with $\gamma_\alpha(n) = U_\alpha\left(\frac{1}{\varepsilon_n}\right)$. T^n is a compact operator for every fixed α and is continuous with respect to α.
Let us suppose by contradiction that $(\alpha_k^n, u_{n,\alpha_k^n})$ is not a bifurcation point and set $F^n(\alpha,v) := v - T^n(\alpha,v)$. Then there exists $\delta_0 > 0$ such that for $\delta \in (0,\delta_0)$ and every $c \in (0,\delta_0)$ we have

$$\begin{aligned} &F^n(\alpha,v) \neq 0, \quad \forall\ \alpha \in (\alpha_k^n-\delta, \alpha_k^n+\delta),\\ &\forall\ v \in \mathcal{H}_n \text{ such that } \|v-u_{n,\alpha}\|_{\mathcal{H}_n} \le c \ \text{ and } v \neq u_{n,\alpha}. \end{aligned} \tag{3.21}$$

We can also choose δ_0 in such a way that the interval $[\alpha_k^n-\delta, \alpha_k^n+\delta]$ does not contain degeneracy points of (3.1) other than α_k^n. Let us consider the set $\Gamma := \{(\alpha,v) \in [\alpha_k^n-\delta,\alpha_k^n+\delta]\times\mathcal{H}_n\ :\ \|v-u_{n,\alpha}\|_{\mathcal{H}_n} < c\}$. Notice that $F^n(\alpha,\cdot)$ is a compact perturbation of the identity and so it makes sense to consider the Leray-Schauder topological degree $deg\,(F^n(\alpha,\cdot),\Gamma_\alpha,0)$ of $F^n(\alpha,\cdot)$ on the set $\Gamma_\alpha := \{v \in \mathcal{H}_n \text{ such that } (\alpha,v)\in\Gamma\}$. From (3.21) it follows that there exist no solutions of $F^n(\alpha,v) = 0$ on $\partial_{[\alpha_k^n-\delta,\alpha_k^n+\delta]\times\mathcal{H}_n}\Gamma$. By the homotopy invariance of the degree, we get

$$deg\,(F^n(\alpha,\cdot),\Gamma_\alpha,0) \text{ is constant on } [\alpha_k^n-\delta,\alpha_k^n+\delta]. \tag{3.22}$$

Since the linearized operator $T_v^n(\alpha,u_{n,\alpha})$ is invertible for $\alpha = \alpha_k^n+\delta$ and $\alpha = \alpha_k^n-\delta$, we have

$$deg\left(F^n(\alpha_k^n-\delta,\cdot),\Gamma_{\alpha_k^n-\delta},0\right) = (-1)^{m(\alpha_k^n-\delta)}$$

and

$$deg\left(F^n(\alpha_k^n+\delta,\cdot),\Gamma_{\alpha_k^n+\delta},0\right) = (-1)^{m(\alpha_k^n+\delta)}.$$

By the choice of α_k^n and of the space $\mathcal{H}_n$ and by (3.20) we get

$$deg\left(F^n(\alpha_k^n-\delta,\cdot),\Gamma_{\alpha_k^n-\delta},0\right) = -deg\left(F^n(\alpha_k^n+\delta,\cdot),\Gamma_{\alpha_k^n+\delta},0\right)$$

contradicting (3.22). Then $(\alpha_k^n, u_{n,\alpha_k^n})$ is a bifurcation point and the bifurcating solutions are nonradial since $u_{n,\alpha}$ is radially nondegenerate for any α as proved in Lemma 3.1. $\square$

Let us observe that these bifurcating solutions lie in the space $\mathcal{H}_n$ and hence are $O(N-1)$-invariant.

Theorem 3.6. *Let $(\alpha_k^n, u_{n,\alpha_k^n})$ be a Morse index changing point related to some k even. Then there exist $[\frac{N}{2}]$ distinct nonradial solutions of* (3.1) *bifurcating from $(\alpha_k^n, u_{n,\alpha_k^n})$.*

Proof. Let us consider the subgroups $\mathcal{G}_h$ of $O(N)$ defined by

$$\mathcal{G}_h = O(h) \times O(N-h) \quad \text{ for } 1 \le h \le \left[\frac{N}{2}\right]. \tag{3.23}$$

In [SW90] (see also [W89]) it is showed that if k is even then the eigenspace V_k of the Laplace-Beltrami operator on S^{N-1}, related to the eigenvalue μ_k invariant by the action of $\mathcal{G}_h$, has dimension one.
Then defining by $\mathcal{H}_n^h$ the subspace of $C_0^{1,\gamma}(\overline{B}_n)$ of functions invariant by the action of $\mathcal{G}_h$ we have that as in (3.20)

$$|m^h(\alpha_k^n + \delta) - m^h(\alpha_k^n - \delta)| = 1 \tag{3.24}$$

if $\delta > 0$ is small enough, where $m^h(\alpha)$ is the Morse index of the radial solution $u_{n,\alpha}$ in the space $\mathcal{H}_n^h$.
Reasoning exactly as in the proof of the previous theorem we get that $(\alpha_k^n, u_{n,\alpha_k^n})$ is a bifurcation point and the bifurcating solution is invariant by the action of $\mathcal{G}_h$.
Moreover, if we get a solution v which is invariant with respect to the action of the two groups $\mathcal{G}_{h_1}$ and $\mathcal{G}_{h_2}$ with $h_1 \ne h_2$, then v must be radial (see [SW90]), and this is not possible, since the radial solutions $u_{n,\alpha}$ are isolated.
Then, we derive the existence of $[\frac{N}{2}]$ distinct nonradial solutions of (3.1) bifurcating from $(\alpha_k^n, u_{n,\alpha_k^n})$. $\square$

Let us denote by Σ_n the closure in $(0,+\infty) \times \mathcal{H}_n$ of the set of solutions of $F^n(\alpha, v) = 0$ different from $u_{n,\alpha}$, i.e

$$\Sigma_n := \overline{\{(\alpha, v) \in (0,+\infty) \times \mathcal{H}_n \, ; \, F^n(\alpha, v) = 0 \, , \, v \ne u_{n,\alpha}\}} \tag{3.25}$$

where $F^n(\alpha, v)$ and $\mathcal{H}_n$ are as in the proof of Theorem 3.5 or 3.6. If $(\alpha_k^n, u_{n,\alpha_k^n}) \in \mathcal{S}(n)$ is a nonradial bifurcation point, then $(\alpha_k^n, u_{n,\alpha_k^n}) \in \Sigma_n$.
For $(\alpha_k^n, u_{n,\alpha_k^n}) \in \Sigma_n$ we will call $\mathcal{C}(\alpha_k^n) \subset \Sigma_n$ the closed connected component of Σ_n which contains $(\alpha_k^n, u_{n,\alpha_k^n})$ and it is maximal with respect to the inclusion.
We have the following:

Proposition 3.7. *Let $(\alpha_k^n, u_{n,\alpha_k^n})$ be a Morse index changing point. If $(\alpha, v_\alpha) \in \mathcal{C}(\alpha_k^n)$ then v_α is a solution of* (3.1) *corresponding to ε_n, in particular $v_\alpha > 0$ in B_n.*

Proof. Let us consider the subset $\mathcal{C} \subset \mathcal{C}(\alpha_k^n)$ of points (α, v_α) which are non-negative solutions of $F^n(\alpha, v) = 0$. Obviously $(\alpha_k^n, u_{n,\alpha_k^n}) \in \mathcal{C}$. We will prove that $\mathcal{C}$ is closed and open in $\mathcal{C}(\alpha_k^n)$, hence $\mathcal{C} = \mathcal{C}(\alpha_k^n)$ since $\mathcal{C}(\alpha_k^n)$ is connected. Moreover, the maximum principle implies that if $(\alpha, v_\alpha) \in \mathcal{C}$ then either $v_\alpha > 0$ or $v_\alpha \equiv 0$, but, since zero is not a solution of (3.1) then the solutions on $\mathcal{C}$ are positive solutions.
If (α, v_α) is a point in the closure of $\mathcal{C}$ then there is a sequence of points (α_j, v_j) in $\mathcal{C}$ that converges to (α, v_α) in $(0, +\infty) \times C_0^{1,\gamma}(\overline{B}_n)$. As $j \to +\infty$ we get that v_α is a solution of $F^n(\alpha, v_\alpha) = 0$ and $v_\alpha \geq 0$ in B_n. By the Maximum principle either $v_\alpha > 0$ or $v_\alpha \equiv 0$ in B_n. But the second case is not possible since zero is not a solution of (3.1). Then $v_\alpha > 0$ in B_n, $(\alpha, v_\alpha) \in \mathcal{C}$ and $\mathcal{C}$ is closed.

Now we will show that $\mathcal{C}$ is open in $\mathcal{C}(\alpha_k^n)$. Let (α, v_α) be a point in $\mathcal{C}$ and $(\bar\alpha, v_{\bar\alpha})$ in $\mathcal{C}(\alpha_k^n)$ such that $\|v_\alpha - v_{\bar\alpha}\|_{\mathcal{H}_n} < \gamma_{\bar\alpha}(n)$, then

$$\begin{aligned} -\Delta v_{\bar\alpha} &= |x|^{\bar\alpha} |v_{\bar\alpha} + \gamma_{\bar\alpha}(n)|^{p_{\bar\alpha}-1}(v_{\bar\alpha} + \gamma_{\bar\alpha}(n)) \\ &= |x|^{\bar\alpha} |v_{\bar\alpha} + \gamma_{\bar\alpha}(n)|^{p_{\bar\alpha}-1}(v_\alpha + v_{\bar\alpha} - v_\alpha + \gamma_{\bar\alpha}(n)) \geq 0 \end{aligned}$$

in B_n and, since $v_{\bar\alpha} = 0$ on ∂B_n, it follows by the maximum principle that $v_{\bar\alpha} > 0$ in B_n.

□

Theorem 3.8. *Let $(\alpha_k^n, u_{n,\alpha_k^n})$ be a Morse index changing point and let $\mathcal{C}(\alpha_k^n)$ be the closed connected component of Σ_n which contains $(\alpha_k^n, u_{n,\alpha_k^n})$ and it is maximal with respect to the inclusion. Then either*

i) $\mathcal{C}(\alpha_k^n)$ is unbounded in $[0, +\infty) \times \mathcal{H}_n$, or

ii) there exists α_h^n with $h \neq k$ such that $(\alpha_h^n, u_{n,\alpha_h^n})$ is a Morse index changing point and $(\alpha_h^n, u_{n,\alpha_h^n}) \in \mathcal{C}(\alpha_k^n)$, or

iii) $\mathcal{C}(\alpha_k^n)$ meets $\{0\} \times \mathcal{H}_n$.

Proof. The proof follows from the global bifurcation result of Rabinowitz, [R71]. One can see also [AM07] or [G10] for details. □

Corollary 3.9. *If $\mathcal{C}(\alpha_k^n)$ is bounded and if it does not meet $\{0\} \times \mathcal{H}_n$ then the number of the Morse index changing points in $\mathcal{C}(\alpha_k^n)$ including $(\alpha_k^n, u_{n,\alpha_k^n})$ is even.*

Proof. The proof follows using an improved version of the Rabinowitz alternative due to Ize and uses the homotopy invariance of the Leray-Schauder degree, see Theorem 3.4.1 in [N01], for details. □

Remark 3.10. The results of Proposition 3.7, Theorem 3.8 and Corollary 3.9 holds for every bifurcation point generated by an odd change in the Morse index of the radial solution $u_{n,\alpha}$. Then, using Theorem 3.6, when k is even we can find $\left[\frac{N}{2}\right]$ different continua of (positive) nonradial solutions bifurcating from $(\alpha_k^n, u_{n,\alpha_k^n})$. Moreover these continua are global in the sense that they satisfy Theorem 3.8.

Remark 3.11. From Remark 3.4 it follows that the Morse index changing exponents α_k^n related to some α_k (i.e. such that $\alpha_k^n \to \alpha_k$ as $n \to +\infty$) for $k > 1$ are in odd number. Moreover Corollary 3.9 says that if $\mathcal{C}(\alpha_k^n)$ is bounded and if it does not interesct $\{0\} \times \mathcal{H}_n$ then the number of the Morse index changing points in $\mathcal{C}(\alpha_k^n)$ including $(\alpha_k^n, u_{n,\alpha_k^n})$ is even.
Then, corresponding to each $k > 1$, we can always choose an exponent $\tilde{\alpha}_k^n$ at which the Morse index of the radial solution $u_{n,\tilde{\alpha}_k^n}$ changes, and such that the continuum $\mathcal{C}(\tilde{\alpha}_k^n)$ is either unbounded or it meets $\{0\} \times \mathcal{H}_n$ or it achieves a Morse index changing point $\tilde{\alpha}_h^n$ which is not contained in $[\alpha_k - \delta, \alpha_k + \delta]$. Hereafter we will call $\tilde{\alpha}_k^n$ the Morse index changing exponents with this property. This will be important in proving the bifurcation result for problem (1.1) in Section 5.

4 Some estimates on the approximating solutions

In this section we give some estimates on the decay of solutions (not necessary radial) of (3.1) as $\varepsilon \to 0$. As before we consider the functions defined in $\mathbb{R}^N$ extended by zero outside of $B_{\frac{1}{\varepsilon}}(0)$, and we denote by $\|\cdot\|_\beta$ and $\|\cdot\|_{1,2}$ the norm of $L^\infty_\beta(\mathbb{R}^N)$ and $D^{1,2}(\mathbb{R}^N)$ respectively. Let X be as defined in (3.7). Then we have the following

Proposition 4.1. *Let ε_n and α_n be sequences such that $\varepsilon_n \to 0$ and $\alpha_n \to \bar{\alpha} > 0$ as $n \to +\infty$. Let v_n be a sequence of solutions of (3.1) in $B_n := B_{\frac{1}{\varepsilon_n}}(0)$, corresponding to the exponent $\alpha = \alpha_n$, i.e.*

$$\begin{cases} -\Delta v_n = C(\alpha_n)|x|^{\alpha_n} (v_n + \gamma(n))^{p_{\alpha_n}} & \text{in } B_n, \\ v_n > 0 & \text{in } B_n, \\ v_n = 0, & \text{on } \partial B_n, \end{cases} \tag{4.1}$$

where $\gamma(n) := U_{\alpha_n}\left(\frac{1}{\varepsilon_n}\right) = \frac{\varepsilon_n^{N-2}}{\left(1+\varepsilon_n^{2+\alpha_n}\right)^{\frac{N-2}{2+\alpha_n}}}$. Assume that $\|v_n\|_X \le A$ for some positive constant A and for every n. Then there exists $C > 0$ such that

$$v_n(x) \le \frac{C}{(1+|x|)^{N-2}} \text{ for every } x \in \mathbb{R}^N \text{ and for every } n \in \mathbb{N}.$$

Proof. We first note that

$$\gamma(n) \le \frac{C}{(1+|x|)^{N-2}} \qquad \forall\ x \in B_n \tag{4.2}$$

and, since $\|v_n\|_\beta \le \|v_n\|_X \le A$ with $\beta < N-2$, we have

$$|v_n(x) + \gamma(n)| \le \frac{C}{(1+|x|)^\beta} \qquad \forall\ x \in B_n. \tag{4.3}$$

We shall use the integral representation of v_n to obtain the desired estimate. If $G_n(x,y)$ denotes the Green function of B_n then

$$\begin{aligned} v_n(x) &= \int_{B_n} G_n(x,y)|y|^{\alpha_n}(v_n(y)+\gamma(n))^{p_{\alpha_n}}\,dy \\ &\le C\int_{B_n} G_n(x,y)\frac{|y|^{\alpha_n}}{(1+|y|)^{\beta p_{\alpha_n}}}dy. \end{aligned} \tag{4.4}$$

Now we consider the function

$$\psi_n(x) = \int_{B_n} G_n(x,y)\frac{|y|^{\alpha_n}}{(1+|y|)^{\beta p_{\alpha_n}}}dy$$

which verifies

$$\begin{cases} -\Delta\psi_n = \frac{|x|^{\alpha_n}}{(1+|x|)^{\beta p_{\alpha_n}}}, & \text{in } B_n \\ \psi_n = 0, & \text{on } \partial B_n. \end{cases}$$

So $\psi_n(r) = \psi_n(|x|)$ satisfies

$$-r^{N-1}\psi_n'(r) = \int_0^r \frac{s^{\alpha_n+N-1}}{(1+s)^{\beta p_{\alpha_n}}}ds. \tag{4.5}$$

Since $\beta > \frac{N}{N+2}(N-2) > \frac{N+\alpha_n}{p_{\alpha_n}}$ for all n, we get that

$$\frac{s^{\alpha_n+N-1}}{(1+s)^{\beta p_{\alpha_n}}} \in L^1(\mathbb{R}_+)$$

and consequently, by (4.5), we have $-\psi_n'(r) \le Cr^{-(N-1)}$. Therefore

$$\psi_n(r) \le \frac{C}{r^{N-2}} \tag{4.6}$$

and, since ψ_n is bounded then $\psi_n(r) \le \frac{C}{(1+r)^{N-2}}$. The claim follows by (4.4) and (4.6). $\square$

Proposition 4.2. *Let ε_n and α_n be sequences such that $\varepsilon_n \to 0$ and $\alpha_n \to \bar{\alpha} > 0$ as $n \to +\infty$. Let v_n be a sequence of solutions of (4.1) in B_n related to the exponents α_n. If $v_n \to U_{\lambda,\bar{\alpha}}$ in X then we have that $\lambda = 1$.*

Proof. By the Pohozaev identity we get

$$\begin{aligned} &-\frac{N\omega_N\gamma(n)^{p_{\alpha_n}+1}}{(p_{\alpha_n}+1)\varepsilon_n^{\alpha_n+N}} + \frac{N-2}{2}\gamma(n)C(\alpha_n)\int_{B_n}|x|^{\alpha_n}(v_n(x)+\gamma(n))^{p_{\alpha_n}}dx \\ &= \frac{1}{2\varepsilon_n}\int_{\partial B_n}\left(\frac{\partial v_n}{\partial\nu}\right)^2 d\sigma_x. \end{aligned} \tag{4.7}$$

Since $v_n \to U_{\lambda,\bar{\alpha}}$ in X, by the standard regularity theory, it follows that

$$v_n \to U_{\lambda,\bar{\alpha}} \quad \text{in } C^2_{loc}(\mathbb{R}^N). \tag{4.8}$$

Recalling that $\gamma(n) = \varepsilon_n^{N-2}(1 + o_n(1))$ we have

$$\frac{\gamma(n)^{p_{\alpha_n}+1}}{\varepsilon_n^{\alpha_n+N}} = O(\varepsilon_n^{(N-2)(p_{\alpha_n}+1)-(\alpha_n+N)}) = O(\varepsilon_n^{N+\alpha_n})$$

and by Proposition 4.1 and (4.8) we derive

$$\int_{B_n} |x|^{\alpha_n}(v_n(x) + \gamma(n))^{p_{\alpha_n}} dx = \int_{\mathbb{R}^N} |x|^{\bar{\alpha}} U_{\lambda,\bar{\alpha}}^{p_{\bar{\alpha}}}(x)dx + o_n(1)$$

Thus (4.7) becomes

$$\varepsilon_n^{N-1}\left((N-2)C(\bar{\alpha})\int_{\mathbb{R}^N} |x|^{\bar{\alpha}} U_{\lambda,\bar{\alpha}}^{p_{\bar{\alpha}}}(x)dx + o_n(1)\right) = \int_{\partial B_n}\left(\frac{\partial v_n}{\partial \nu}\right)^2 d\sigma_x. \quad (4.9)$$

Next we expand the right-hand side of (4.9) using the Green function G_n of B_n. For $x \in \partial B_n$, we write

$$\frac{\partial v_n}{\partial \nu}(x) = C(\alpha_n)\int_{B_n} \frac{\partial G_n}{\partial \nu_x}(x, y)|y|^{\alpha_n}(v_n(y) + \gamma(n))^{p_{\alpha_n}} dy \quad (4.10)$$

and substituting (here ω_N denotes the volume of the $N-$dimensional unit ball)

$$\frac{\partial G_n}{\partial \nu_x}(x, y) = -\frac{1}{N\omega_N}\frac{1 - \varepsilon_n^2|y|^2}{\varepsilon_n|x - y|^N}$$

into (4.10) gives

$$\begin{aligned}\frac{\partial v_n}{\partial \nu}(x) &= -(C(\bar{\alpha}) + o_n(1))\int_{B_n} \frac{1}{N\omega_N}\frac{1 - \varepsilon_n^2|y|^2}{\varepsilon_n|x - y|^N}|y|^{\alpha_n}(v_n(y) + \gamma(n))^{p_{\alpha_n}} dy \\ &= -\frac{\varepsilon_n^{N-1}(C(\bar{\alpha}) + o_n(1))}{N\omega_N}\int_{B_n} \frac{(1 - \varepsilon_n^2|y|^2)|y|^{\alpha_n}}{|z - \varepsilon_n y|^N}(v_n(y) + \gamma(n))^{p_{\alpha_n}} dy\end{aligned} \quad (4.11)$$

where we have posed $x = \frac{z}{\varepsilon_n}$, $z \in S^{N-1}$. Let us assume that we have proved

$$\int_{B_n} \frac{(1 - \varepsilon_n^2|y|^2)|y|^{\alpha_n}}{|z - \varepsilon_n y|^N}(v_n(y) + \gamma(n))^{p_{\alpha_n}} dy = \int_{\mathbb{R}^N} |y|^{\bar{\alpha}} U_{\lambda,\bar{\alpha}}^{p_{\bar{\alpha}}}(y)dy + o_n(1) \quad (4.12)$$

uniformly with respect to $z \in S^{N-1}$, then (4.11) becomes

$$\frac{\partial v_n}{\partial \nu}(x) = -\frac{\varepsilon_n^{N-1}C(\bar{\alpha})}{N\omega_N}\left(\int_{\mathbb{R}^N} |y|^{\bar{\alpha}} U_{\lambda,\bar{\alpha}}^{p_{\bar{\alpha}}}(y)dy + o_n(1)\right) \quad (4.13)$$

where the term $o_n(1)$ is uniform in $x \in \partial B_n$. Thus

$$\left(\frac{\partial v_n}{\partial \nu}(x)\right)^2 = \frac{\varepsilon_n^{2(N-1)}(C(\bar{\alpha}))^2}{N^2\omega_N^2}\left[\left(\int_{\mathbb{R}^N} |y|^{\bar{\alpha}} U_{\lambda,\bar{\alpha}}^{p_{\bar{\alpha}}}(y)dy\right)^2 + o_n(1)\right]$$

and

$$\int_{\partial B_n}\left(\frac{\partial v_n}{\partial \nu}(x)\right)^2 d\sigma_x = \frac{\varepsilon_n^{(N-1)}\left(C(\bar\alpha)\right)^2}{N\omega_N}\left[\left(\int_{{\rm I\!R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy\right)^2 + o_n(1)\right].$$

Turning back to (4.9) we have

$$(N-2)\int_{{\rm I\!R}^N}|x|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(x)dx + o_n(1) = \frac{C(\bar\alpha)}{N\omega_N}\left(\int_{{\rm I\!R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy\right)^2$$

and passing to the limit

$$(N-2)\int_{{\rm I\!R}^N}|x|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(x)dx = \frac{C(\bar\alpha)}{N\omega_N}\left(\int_{{\rm I\!R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy\right)^2 \qquad \Longrightarrow$$

$$(N-2) = \frac{C(\bar\alpha)}{N\omega_N}\int_{{\rm I\!R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy. \tag{4.14}$$

A straightforward computation gives

$$\int_{{\rm I\!R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy = \frac{N\omega_N}{\lambda^{\frac{N-2}{2}}}\frac{1}{N+\bar\alpha} \tag{4.15}$$

Then, by (4.14) and (4.15), we infer

$$N-2 = \frac{C(\bar\alpha)}{\lambda^{\frac{N-2}{2}}}\frac{1}{N+\bar\alpha} = \frac{N-2}{\lambda^{\frac{N-2}{2}}} \qquad \Longrightarrow \qquad \lambda = 1. \tag{4.16}$$

which gives the claim.
It remains to verify (4.12), which is a straightforward calculation. Using the decay of v_n and the Lebesgue's dominated convergence theorem we get

$$\int_{|y|\le\frac{1}{2\varepsilon_n}}\frac{(1-\varepsilon_n^2|y|^2)|y|^{\alpha_n}}{|z-\varepsilon_n y|^N}(v_n(y)+\gamma(n))^{p_{\alpha_n}}dy \quad \longrightarrow \quad \int_{{\rm I\!R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy$$

uniformly with respect to z as $\varepsilon_n \to 0$, recalling that $|z| = 1$. Finally we estimate the integral in the rest of the domain $\frac{1}{2\varepsilon_n} \le |y| \le \frac{1}{\varepsilon_n}$. Using again the decay of v_n we have

$$\begin{aligned}
&\int_{\frac{1}{2\varepsilon_n}\le|y|\le\frac{1}{\varepsilon_n}}\frac{(1-\varepsilon_n^2|y|^2)|y|^{\alpha_n}}{|z-\varepsilon_n y|^N}(v_n(y)+\gamma(n))^{p_{\alpha_n}}dy \\
&\qquad \le C\varepsilon_n^{N+2+\alpha_n}\int_{\frac{1}{2\varepsilon_n}\le|y|\le\frac{1}{\varepsilon_n}}\frac{1-\varepsilon_n^2|y|^2}{|z-\varepsilon_n y|^N}dy \\
&\qquad \le C\varepsilon_n^{2+\alpha_n}\int_{\frac{1}{2}\le|\xi-z|\le 1}\frac{1-|z-\xi|^2}{|\xi|^N}d\xi \\
&\qquad \le C\varepsilon_n^{2+\alpha_n}\int_{\frac{1}{2}\le|\xi-z|\le 1}\frac{2+|\xi|}{|\xi|^{N-1}}d\xi \quad \longrightarrow \quad 0, \qquad \text{as} \quad \varepsilon_n \to 0,
\end{aligned}$$

uniformly with respect to z. The proof is now complete. □

Lemma 4.3. *Let ε_n and α_n be sequences such that $\varepsilon_n \to 0$ and $\alpha_n \to \bar{\alpha} > 0$ as $n \to +\infty$. Let v_n be a sequence of nonradial solutions of* (4.1) *in B_n related to the exponents α_n. If $v_n \to U_{\bar{\alpha}}$ in X then there exists a constant $C > 0$ such that*

$$\frac{|u_n - v_n|}{\|u_n - v_n\|_\infty} \leq \frac{C}{(1+|x|)^{N-2}} \tag{4.17}$$

for any n sufficiently large, where u_n is the radial solution of (4.1) *(corresponding to $\alpha = \alpha_n$) as defined in* (3.2)*.*

Proof. We let $w_n := \frac{u_n - v_n}{\|u_n - v_n\|_\infty}$. Then w_n satisfies

$$\begin{cases} -\Delta w_n = C(\alpha_n)|x|^{\alpha_n} a_n(x) w_n & \text{in } B_n \\ w_n = 0 & \text{on } \partial B_n \end{cases} \tag{4.18}$$

where $a_n(x) = p_n \int_0^1 (tu_n + (1-t)v_n + \gamma(n))^{p_n - 1}\, dt$, for $p_n := p_{\alpha_n}$ and $\gamma(n)$ as before. By hypothesis $v_n \to U_{\bar{\alpha}}$ in X and, by Proposition 4.1 and Proposition 3.2, there exists a constant $C > 0$ (independent on n), such that

$$|v_n(x)| \leq \frac{C}{(1+|x|)^{N-2}} \quad \text{and} \quad |u_n(x)| \leq \frac{C}{(1+|x|)^{N-2}}.$$

Then, by (4.2), we have

$$\begin{aligned} |a_n(x)| &\leq C\left(|u_n|^{p_n-1} + |v_n|^{p_n-1} + (\gamma(n))^{p_n-1}\right) \\ &\leq C\left(\frac{1}{(1+|x|)^{(N-2)(p_n-1)}}\right) \\ &\leq \frac{C}{(1+|x|)^{4+2\alpha_n}}. \end{aligned} \tag{4.19}$$

We consider the Kelvin transform of w_n, i.e.

$$\widehat{w}_n(x) := \frac{1}{|x|^{N-2}} w_n\left(\frac{x}{|x|^2}\right), \quad x \in \mathbb{R}^N \setminus B_{\varepsilon_n}.$$

It satisfies

$$\begin{cases} -\Delta \widehat{w}_n = C(\alpha_n)\frac{1}{|x|^{4+\alpha_n}} a_n(\frac{x}{|x|^2})\widehat{w}_n & \text{in } R^N \setminus B_{\varepsilon_n} \\ \widehat{w}_n = 0 & \text{on } \partial B_{\varepsilon_n} \end{cases}$$

and, using (4.19),

$$\frac{1}{|x|^{4+\alpha_n}} a_n\left(\frac{x}{|x|^2}\right) \leq \frac{1}{|x|^{4+\alpha_n}} \frac{C}{\left(1+\frac{1}{|x|}\right)^{4+2\alpha_n}} = \frac{|x|^{\alpha_n}}{(1+|x|)^{4+2\alpha_n}}.$$

Then, since $\widehat{w}_n = 0$ on $\partial B_{\varepsilon_n}$, the regularity theory (see [H91]) implies

$$\|\widehat{w}_n\|_{L^\infty(B_1 \setminus B_{\varepsilon_n})} \leq C\|\widehat{w}_n\|_{L^{2^*}(B_2 \setminus B_{\varepsilon_n})}. \tag{4.20}$$

We will show that there exists a constant C (independent on n) such that $\|\widehat{w}_n\|_{L^{2^*}(B_2\setminus B_{\varepsilon_n})} \leq C$ and then (4.20) will imply (4.17).
Using the Sobolev embedding Theorem, (4.18) and (4.19) we have

$$\begin{aligned}\|w_n\|^2_{L^{2^*}(B_n)} &\leq \frac{1}{S}\int_{B_n} |\nabla w_n|^2\,dx = \frac{C(\alpha_n)}{S}\int_{B_n} |x|^{\alpha_n} a_n(x) w_n^2\,dx \\ &\leq C\int_{B_n} \frac{1}{(1+|x|)^{4+\alpha_n}}|w_n|^2\,dx \leq C\int_{B_n} \frac{1}{(1+|x|)^{4+\alpha_n}}|w_n|^{2-\delta}\,dx\end{aligned}$$

since $|w_n| \leq 1$, for some $\delta > 0$ that we will choose later. Now, using the Hölder inequality, we get

$$\begin{aligned}\|w_n\|^2_{L^{2^*}(B_n)} &\leq \left(\int_{B_n}\left(\frac{1}{(1+|x|)^{4+\alpha_n}}\right)^{\frac{2N}{4+\delta(N-2)}} dx\right)^{\frac{4+\delta(N-2)}{2N}} \left(\int_{B_n} |w_n|^{2^*}\,dx\right)^{\frac{2-\delta}{2^*}} \\ &\leq C_\delta \left(\int_{B_n} |w_n|^{2^*}\,dx\right)^{\frac{2-\delta}{2^*}} = C_\delta \|w_n\|^{2-\delta}_{L^{2^*}(B_n)}\end{aligned}$$

if $0 < \delta < \min\{2, \frac{4+\bar{\alpha}}{N-2}\}$ where $\bar{\alpha} = \lim \alpha_n$. Note that the constant C_δ is independent on n, for n large enough, and using (4.20) we get

$$\|\widehat{w}_n\|_{L^\infty(B_1\setminus B_{\varepsilon_n})} \leq C\|\widehat{w}_n\|_{L^{2^*}(\mathbb{R}^N\setminus B_{\varepsilon_n})} = C\|w_n\|_{L^{2^*}(B_n)} \leq C.$$

This concludes the proof. □

Proposition 4.4. *Let ε_n and α_n be a sequences such that $\varepsilon_n \to 0$ and $\alpha_n \to \bar{\alpha} > 0$ as $n \to +\infty$. Let v_n be a sequence of nonradial solutions of* (4.1) *in B_n related to the exponents α_n. If $\bar{\alpha} \neq \alpha_k$ for all $k \in \mathbb{N}$ then there exists a constant $c > 0$ (independent on n), such that*

$$\|v_n - u_n\|_\infty \geq c \tag{4.21}$$

where u_n is the radial solution of (4.1) *in B_n (corresponding to the exponents α_n) as defined in* (3.2).

Proof. Let us suppose, by contradiction, that there exists a sequence of nonradial solutions v_n of (4.1) in B_n related to the exponent α_n such that

$$\|v_n - u_n\|_\infty \to 0 \qquad \text{as } n \to +\infty. \tag{4.22}$$

Both solutions satisfy the Pohozaev identity (4.7), so if we write the identities for u_n and v_n and subtract one from another we obtain

$$\begin{aligned}(N-2)\varepsilon_n\gamma(n)C(\alpha_n)\int_{B_n} |x|^{\alpha_n}\left[(u_n(x)+\gamma(n))^{p_{\alpha_n}} - (v_n(x)+\gamma(n))^{p_{\alpha_n}}\right]dx \\ = \int_{\partial B_n} \frac{\partial}{\partial\nu}(u_n(x)-v_n(x))\frac{\partial}{\partial\nu}(u_n(x)+v_n(x))d\sigma_x.\end{aligned} \tag{4.23}$$

By mean value theorem we get

$$
\begin{aligned}
&(u_n(x)+\gamma(n))^{p_{\alpha_n}} - (v_n(x)+\gamma(n))^{p_{\alpha_n}} \\
&\quad = p_{\alpha_n}\int_0^1 (tu_n(x)+(1-t)v_n(x)+\gamma(n))^{p_{\alpha_n}-1}\,dt\ (u_n(x)-v_n(x)) \\
&\quad = a_n(x)(u_n(x)-v_n(x))
\end{aligned}
$$

and setting $w_n = \frac{u_n - v_n}{\|v_n-u_n\|_\infty}$ (4.23) becomes

$$
(N-2)\varepsilon_n\gamma(n)C(\alpha_n)\int_{B_n}|x|^{\alpha_n}a_n(x)w_n(x)dx = \int_{\partial B_n}\frac{\partial w_n}{\partial\nu}\frac{\partial}{\partial\nu}(u_n(x)+v_n(x))d\sigma_x. \tag{4.24}
$$

Note that, by (3.9) and (4.22), we have that

$$
a_n \to p_{\bar\alpha}U_{\bar\alpha}^{p_{\bar\alpha}-1} \qquad \text{in} \quad C^2_{loc}(\mathbb{R}^N),
$$

and the function w_n satisfies the following equation

$$
\begin{cases} -\Delta w_n = C(\alpha_n)|x|^{\alpha_n}a_n(x)w_n, & \text{in } \ B_n \\ w_n = 0, & \text{in } \ \partial B_n. \end{cases} \tag{4.25}
$$

Since $\|w_n\|_\infty = 1$, by standard regularity theorems, one can prove that $w_n \to w$ in $C^2_{loc}(\mathbb{R}^N)$ where w satisfies

$$
\begin{cases} -\Delta w = C(\bar\alpha)p_{\bar\alpha}|x|^{\bar\alpha}U_{\bar\alpha}^{p_{\bar\alpha}-1}w, & \text{in } \ \mathbb{R}^N \\ |w|\le 1, & \text{in } \ \mathbb{R}^N. \end{cases} \tag{4.26}
$$

and, because $\bar\alpha \neq \alpha_k$, by Theorem 1.3 we have

$$
w(x) = A\frac{1-|x|^{2+\bar\alpha}}{(1+|x|^{2+\bar\alpha})^{\frac{N+\bar\alpha}{2+\bar\alpha}}} \qquad \text{for some} \quad A\in\mathbb{R}. \tag{4.27}
$$

The next step is to prove that $A = 0$. In order to do this, we shall expand both sides of (4.24) in powers of ε_n as we did in the proof of Proposition 4.2 and then pass to the limit.

Let us expand first the LHS of (4.24). Using the decay properties of w_n, u_n, v_n and Lebesgue's theorem we have

$$
\int_{B_n}|x|^{\alpha_n}a_n(x)w_n(x)dx = p_{\bar\alpha}\int_{\mathbb{R}^N}|x|^{\bar\alpha}w(x)U_{\bar\alpha}^{p_{\bar\alpha}-1}(x)dx + o_n(1).
$$

Hence the LHS of (4.24) becomes

$$
LHS = (N-2)C(\bar\alpha)\varepsilon_n^{N-1}\left(p_{\bar\alpha}\int_{\mathbb{R}^N}|x|^{\bar\alpha}w(x)U_{\bar\alpha}^{p_{\bar\alpha}-1}(x)dx + o_n(1)\right) \tag{4.28}
$$

Now, we expand the RHS of (4.24). For $x \in \partial B_n$, as in (4.10)-(4.13), we can write

$$\begin{aligned}\frac{\partial w_n}{\partial \nu}(x) &= C(\alpha_n)\int_{B_n}\frac{\partial G_n}{\partial \nu_x}(x,y)|y|^{\alpha_n}a_n(y)w_n(y)dy \\ &\quad \big(\text{setting } z=\varepsilon_n x,\ \ z\in S^{N-1}\big) \\ &= -\frac{\varepsilon_n^{N-1}C(\alpha_n)}{N\omega_N}\int_{B_n}\frac{(1-\varepsilon_n^2|y|^2)|y|^{\alpha_n}}{|z-\varepsilon_n y|^N}a_n(y)w_n(y)dy \\ &= -\frac{\varepsilon_n^{N-1}C(\bar\alpha)}{N\omega_N}\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}p_{\bar\alpha}U_{\bar\alpha}^{p_{\bar\alpha}-1}(y)w(y)dy+o_n(1)\right).\end{aligned} \tag{4.29}$$

Also, by (4.13), one has

$$\frac{\partial u_n}{\partial \nu}(x)+\frac{\partial v_n}{\partial \nu}(x) = -2\frac{\varepsilon_n^{N-1}C(\bar\alpha)}{N\omega_N}\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy+o_n(1)\right) \tag{4.30}$$

Hence, by (4.29) and (4.30) we obtain

$$\begin{aligned}&\int_{\partial B_n}\frac{\partial w_n}{\partial \nu}\frac{\partial}{\partial \nu}(u_n(x)+v_n(x))d\sigma_x \\ &= 2\frac{\varepsilon_n^{2(N-1)}\left(C(\bar\alpha)\right)^2}{N^2\omega_N^2}\Bigg[\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}p_{\bar\alpha}U_{\bar\alpha}^{p_{\bar\alpha}-1}(y)w(y)dy\right)\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy\right) \\ &\qquad\qquad +o_n(1)\Bigg]|\partial B_n| \\ &= 2\frac{\varepsilon_n^{(N-1)}\left(C(\bar\alpha)\right)^2}{N\omega_N}\Bigg[\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}p_{\bar\alpha}U_{\bar\alpha}^{p_{\bar\alpha}-1}(y)w(y)dy\right)\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy\right) \\ &\qquad\qquad +o_n(1)\Bigg].\end{aligned} \tag{4.31}$$

Therefore, substituting (4.28), (4.29) and (4.30) into (4.24), canceling the terms which appear on both sides and passing to the limit we get

$$\begin{aligned}&(N-2)\int_{\mathbb{R}^N}|x|^{\bar\alpha}w(x)U_{\bar\alpha}^{p_{\bar\alpha}-1}(x)dx \\ &= 2\frac{C(\bar\alpha)}{N\omega_N}\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}U_{\bar\alpha}^{p_{\bar\alpha}-1}(y)w(y)dy\right)\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy\right)\end{aligned}$$

and using (4.27) we get,

$$\begin{aligned}&A(N-2)\int_{\mathbb{R}^N}|x|^{\bar\alpha}\frac{1-|x|^{2+\bar\alpha}}{(1+|x|^{2+\bar\alpha})^{\frac{N+\bar\alpha}{2+\bar\alpha}}}U_{\bar\alpha}^{p_{\bar\alpha}-1}(x)dx \\ &= 2\frac{AC(\bar\alpha)}{N\omega_N}\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}\frac{1-|y|^{2+\bar\alpha}}{(1+|y|^{2+\bar\alpha})^{\frac{N+\bar\alpha}{2+\bar\alpha}}}U_{\bar\alpha}^{p_{\bar\alpha}-1}(y)dy\right)\left(\int_{\mathbb{R}^N}|y|^{\bar\alpha}U_{\lambda,\bar\alpha}^{p_{\bar\alpha}}(y)dy\right).\end{aligned} \tag{4.32}$$

One can verify that

$$\int_{\mathbb{R}^N} |y|^{\bar\alpha} \frac{1-|y|^{2+\bar\alpha}}{(1+|y|^{2+\bar\alpha})^{\frac{N+\bar\alpha}{2+\bar\alpha}}} U_{\bar\alpha}^{p_{\bar\alpha}-1}(y)dy = -\frac{N\omega_N(N-2)}{C(\bar\alpha)p_{\bar\alpha}} \neq 0 \tag{4.33}$$

and by (4.15) we deduce

$$A(N-2) = 2\frac{AC(\bar\alpha)}{N\omega_N}\frac{N\omega_N}{N+\bar\alpha} = 2A(N-2) \quad \Longrightarrow \quad A = 0. \tag{4.34}$$

Therefore

$$w_n \to 0 \quad \text{in } C^2_{loc}(\mathbb{R}^N). \tag{4.35}$$

Let $x_n \in B_n$ be such that $|w_n(x_n)| = 1 = \|w_n\|_\infty$. By (4.17) the sequence x_n remains bounded, but this contradicts (4.35).
So (4.22) cannot occur and this gives the claim. $\square$

5 The bifurcation result

Let us consider the radial solution U_α of problem (1.1) for $\alpha \in (0,+\infty)$. As shown in Section 2 the solutions U_α are always degenerate in the space of radial functions. Indeed the linearized equation (1.9) has the radial solution $Z(x)$, as in (1.10) for any $\alpha \in (0,+\infty)$. On the other hand, when α is even, the kernel of the linearized operator is richer and it is generated by the functions in (1.11).
Moreover, as shown in Corollary 1.4 the Morse index of U_α changes as α crosses α_k, with $\alpha_k = 2(k-1)$ and all the eigenfunctions associated to the linearized problem, (i.e. the solutions of (2.11)) lie in the space X, defined in (3.7). Let us define the space

$$\mathcal{H} := \{v \in X \text{ s.t. } v(x_1,\dots,x_N) = v(g(x_1,\dots,x_{N-1}),x_N),\ \forall\, g \in O(N-1)\}.$$

Using a result of [SW86] we have that the Morse index of U_α in α_k increases by one if we restrict to the space $\mathcal{H}$. Then we get

$$m(\alpha_k+\delta) - m(\alpha_k-\delta) = 1$$

if m is the Morse index of U_α in the space $\mathcal{H}$. We want to use this change in the Morse index of U_α (in the space $\mathcal{H}$) to prove the existence of continua of nonradial solutions of (1.1) bifurcating from (α_k, U_{α_k}) in the product space $(0,+\infty)\times\mathcal{H}$.
But, due to the degeneracy of the radial solution U_α for any α, we cannot obtain the bifurcation result directly. Then to get the desired result we consider the approximating problem (3.1).

5.1 Proof of the main theorem

Given a sequence $\varepsilon_n \to 0$, we have a sequence of nondegenerate radial solutions $u_{n,\alpha}$ of (3.1) (corresponding to $\varepsilon = \varepsilon_n$), that converges to U_α as $n \to +\infty$.

In Section 3.1, 3.2 we proved that $(\tilde{\alpha}_k^n, u_{n,\tilde{\alpha}_k^n})$ are nonradial bifurcation points and give rise to the continua $\mathcal{C}(\tilde{\alpha}_k^n)$ in the space $(0,+\infty)\times\mathcal{H}_n$ (but also in the space $(0,+\infty)\times\mathcal{H}_n^h$, if k is even, where $\mathcal{H}_n^h$ is as in the proof of Theorem 3.6). These continua $\mathcal{C}(\tilde{\alpha}_k^n)$ are global and obey the so called Rabinowitz alternative Theorem, (see Theorem 3.8 and Remark 3.11).
In this section we want to prove that these continua $\mathcal{C}(\tilde{\alpha}_k^n)$ converge in a suitable sense, as $n\to+\infty$ to continua of nonradial solutions of (1.1) that bifurcate from (α_k, U_{α_k}). To do this we use some ideas already used in [AG97], see also [GP11]. Extending the functions by zero outside of B_n and by regularity theorems we can infer that $\mathcal{C}(\tilde{\alpha}_k^n)$ belongs to the space

$$Z := (0,+\infty)\times\mathcal{H},$$

where $\mathcal{H}$ is as defined before. Moreover, by Proposition 3.2 $u_{n,\tilde{\alpha}_k^n}\to U_{\alpha_k}$ in $\mathcal{H}\subset X$ as $n\to+\infty$.
To prove the bifurcation result we need the following topological lemma (see Lemma 3.1 in [AG97]).

Lemma 5.1. *Let X_n be a sequence of connected subsets of a metric space X. If*

(i) $\liminf (X_n)\neq\emptyset$;

(ii) $\bigcup X_n$ *is precompact;*

then $\limsup (X_n)$ *is nonempty, compact and connected.*

Above, $\liminf (X_n)$ and $\limsup (X_n)$ denote the set of all $x\in X$ such that any neighborhood of x intersects all but finitely many of X_n, infinitely many of X_n respectively.

Now let us fix $k>1$ and $\alpha_k = 2(k-1)$. Take n sufficiently large and let $(\tilde{\alpha}_k^n, u_{n,\tilde{\alpha}_k^n})$ be the bifurcation point for problem (3.1) as in Remark 3.11. For simplicity we set $\alpha_n := \tilde{\alpha}_k^n$ and $u_n := u_{n,\tilde{\alpha}_k^n}$. Let $\mathcal{C}(\alpha_n)$ be the maximal connected component which bifurcates from (α_n, u_n) in the space Z. Let $\delta>0$ such that in the interval $[\alpha_k-2\delta, \alpha_k+2\delta]$ there is not another exponent α_h with $h\neq k$. We let $\mathcal{Z}_n := \mathcal{C}(\alpha_n)\cap B_{\delta,X}(\alpha_n,u_n)$ where

$$B_{\delta,Z}(\alpha_n,u_n) := \{(\alpha,v)\in Z \text{ such that } |\alpha-\alpha_n| + \|v-u_n\|_X < \delta\}$$

where the space X and its norm are as defined in (3.7) and (3.8). Finally we denote by $\mathcal{X}_n$ the maximal connected component of $\mathcal{Z}_n$ that contains (α_n,u_n).

Remark 5.2. *The sets $\mathcal{X}_n$ are nonempty since they contain at least (α_n,u_n), moreover $(\alpha_k,U_{\alpha_k})\in\liminf(\mathcal{X}_n)$.*

We have

Lemma 5.3. *The set $\bigcup\mathcal{X}_n$ is precompact in Z.*

Proof. Let (α_m, v_m) be a sequence in $\bigcup \mathcal{X}_n$. Then $(\alpha_m, v_m) \in \mathcal{X}_{n(m)}$ for some $n(m) > 0$. We consider first the case where $n(m) \to +\infty$ as $m \to \infty$.
By the definition of $\mathcal{X}_{n(m)}$, the functions v_m satisfy (3.1) with $\varepsilon = \varepsilon_{n(m)}$ and $\alpha = \alpha_m$. Then $\alpha_m \in (\alpha_k - 2\delta, \alpha_k + 2\delta)$ since $|\alpha_m - \alpha_k| \le |\alpha_m - \alpha_{n(m)}| + |\alpha_{n(m)} - \alpha_k| \le 2\delta$ if $n(m)$ is large enough. Hence, up to a subsequence $\alpha_{(m)} \to \bar{\alpha}$ and $\bar{\alpha} \in [\alpha_k - 2\delta, \alpha_k + 2\delta]$. Moreover $v_m \in \mathcal{H}$ and $\|v_m - u_{n(m)}\|_X < \delta$ so that $\|v_m\|_X < \delta + \sup_m \|u_{n(m)}\|_X$. From (3.9) we have that $\|u_{n(m)}\|_X \le C$. Then $\|v_m\|_{1,2} \le A$ and $\|v_m\|_\beta \le A$ for any m for some constant $A > 0$. Up to a subsequence $v_m \to \bar{v}$ weakly in $D^{1,2}(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$. Moreover from (3.1) we get that $v_m \to \bar{v}$ in $C^1_{loc}(\mathbb{R}^N)$, where $\bar{v}$ is a solution of (1.1) with $\alpha = \bar{\alpha}$.
Further from Proposition 4.1 there exists $C > 0$, independent of m, such that $v_m(x) \le \frac{C}{(1+|x|)^{N-2}}$. Thus

$$\int_{\mathbb{R}^N} |\nabla v_m|^2\, dx = C(\alpha_m) \int_{B_m} |x|^{\alpha_m} \left(v_m + \gamma(m)\right)^{p_m} v_m\, dx$$

where $\gamma(m) := \gamma_{\alpha_m}(\varepsilon_m)$ and $p_m := p_{\alpha_m}$ and passing to the limit, using $v_m + \gamma(m) \le \frac{C}{(1+|x|)^{N-2}}$, (see (4.2)),

$$\int_{\mathbb{R}^N} |\nabla v_m|^2\, dx \to C(\bar{\alpha}) \int_{\mathbb{R}^N} |x|^{\bar{\alpha}} \bar{v}^{p_{\bar{\alpha}}+1}\, dx = \int_{\mathbb{R}^N} |\nabla \bar{v}|^2\, dx.$$

Hence

$$\int_{\mathbb{R}^N} |\nabla \left(v_m - \bar{v}\right)|^2\, dx = \int_{\mathbb{R}^N} |\nabla v_m|^2\, dx - 2\int_{\mathbb{R}^N} \nabla v_m \cdot \nabla \bar{v}\, dx + \int_{\mathbb{R}^N} |\nabla \bar{v}|^2\, dx \to 0$$

as $m \to \infty$ so that $v_m \to v$ strongly in $D^{1,2}(\mathbb{R}^N)$.
Moreover from Proposition 4.1 we have that

$$|v_m(x) - \bar{v}(x)| \le |v_m(x)| + |\bar{v}(x)| \le \frac{C}{(1+|x|)^{N-2}}.$$

This means that for every $\varepsilon > 0$ there exists $\rho > 0$ such that, for any m, $(1+|x|)^\beta |v_m(x) - \bar{v}(x)| < \varepsilon$ if $|x| > \rho$. Then from the uniform convergence of v_m to $\bar{v}$ on compact sets of $\mathbb{R}^N$ we have that $(1+|x|)^\beta |v_m(x) - \bar{v}(x)| < \varepsilon$ in $B_\rho(0)$ if m is sufficiently large, i.e $v_m \to \bar{v}$ in $L^\infty_\beta(\mathbb{R}^N)$. Then $v_m \to \bar{v}$ strongly in X.
If the sequence $n(m) \nrightarrow +\infty$, up to a subsequence $n(m)$ converges to $n_0 \in \mathbb{N}$ and repeating the proof for the first case we have that a subsequence v_{m_k} converges in X to a solution of (3.1) for $\alpha = \bar{\alpha}$ and $\varepsilon = \varepsilon_{n_0}$. □

Lemma 5.4. *The set* $\limsup (\mathcal{X}_n) \setminus \{(\alpha_k, U_{\alpha_k})\}$ *is nonempty.*

Proof. By the results of Section 3.2 (Theorem 3.8, Corollary 3.9 and Remark 3.11) and regularity theorems a global continuum bifurcates from the point (α_n, u_n) and this continuum $\mathcal{C}(\alpha_n)$ is either unbounded in Z or it meets $\{0\} \times$

X or it achieves a Morse index changing point $\tilde{\alpha}_h^n$ which is not contained in $[\alpha_k - \delta, \alpha_k + \delta]$ (see Remark 3.11). This implies that, on the closure of any component $\mathcal{X}_n$, there exists a point $(\bar{\alpha}_n, \bar{v}_n) \in \partial B_{\delta,Z}(\alpha_n, u_n)$ i.e. such that

$$|\bar{\alpha}_n - \alpha_n| + \|\bar{v}_n - u_n\|_X = \delta \tag{5.1}$$

and $\bar{v}_n$ is a solution of (3.1) in B_n for $\alpha = \bar{\alpha}_n$.
Using the bounds on $\bar{v}_n$ and $\bar{\alpha}_n$ and the standard regularity theorems we can pass to the limit and get that $(\bar{\alpha}_n, \bar{v}_n) \to (\bar{\alpha}, \bar{v})$ with $\bar{v}$ solution of (1.1) for $\alpha = \bar{\alpha}$, $\bar{\alpha} \in [\alpha_k - 2\delta, \alpha_k + 2\delta]$ and

$$|\bar{\alpha} - \alpha_k| + \|\bar{v} - U_{\alpha_k}\|_X = \delta.$$

Then, obviously, $(\bar{\alpha}, \bar{v}) \in \limsup(\mathcal{X}_n)$ but $(\bar{\alpha}, \bar{v}) \neq (\alpha_k, U_{\alpha_k})$. □

Theorem 5.5. *For any $k \geq 2$, the points (α_k, U_{α_k}) are nonradial bifurcation points for the curve $\mathcal{S}$ of radial solution of* (1.1)*, i.e.*

$$\mathcal{S} := \left\{ \begin{array}{c} (\alpha, U_\alpha) \in (0, +\infty) \times X \ \ \textit{such that} \ \ U_\alpha \ \ \textit{is the} \\ \textit{unique radial solution of } (1.1) \ \textit{such that} \ \ U_\alpha(0) = 1 \end{array} \right\} \tag{5.2}$$

and X as defined in (3.7)*.*

Proof. Let $\alpha_k = 2(k-1)$. Then, fixing k as before, we consider the bifurcation points (α_n, u_n) for problem (3.1) in B_n and the connected components $\mathcal{X}_n$ of the bifurcation continua in $B_{\delta,Z}(\alpha_n, u_n)$.
By Remark 5.2 and Proposition 5.3 the sequence of sets $\mathcal{X}_n$ satisfies the hypotheses of Lemma 5.1 in the space Z and hence the set

$$\mathcal{C}_k = \limsup(\mathcal{X}_n)$$

is nonempty, compact and connected. Moreover it contains (α_k, U_{α_k}) and it does not reduce only to this point by Lemma 5.4.
If $(\bar{\alpha}, \bar{v}) \in \mathcal{C}_k \setminus (\alpha_k, U_{\alpha_k})$ by definition there exists a sequence of points $(\bar{\alpha}_n, v_n) \in \mathcal{X}_n$ such that $(\bar{\alpha}_n, v_n) \to (\bar{\alpha}, \bar{v})$ in Z and $\bar{v}$ is a solution of (1.1) for $\alpha = \bar{\alpha}$ and $\bar{v} > 0$ because $(\bar{\alpha}_n, v_n) \in B_{\delta,Z}(\alpha_n, u_n)$ and δ is small. We want to show that $\bar{v}$ is a nonradial solution of (1.1) for $\alpha = \bar{\alpha}$. To this end we need to show that $\bar{v} \neq U_{\lambda,\bar{\alpha}}$ for any $\lambda > 0$.
From Proposition 4.2 we have that $\bar{v} \neq U_{\lambda,\bar{\alpha}}$ for any $\lambda \neq 1$. This implies, in turn, that $\bar{v}$ is a nonradial solution of (1.1) if $\bar{\alpha} = \alpha_k$ since we suppose $(\bar{\alpha}, \bar{v}) \in \mathcal{C}_k \setminus (\alpha_k, U_{\alpha_k})$.
Then, the claim follows by showing that

$$\|v_n - U_{\bar{\alpha}}\|_X > c > 0$$

for some positive constant c and for any n sufficiently large, for $\bar{\alpha} \neq \alpha_k$. Equivalently we will show that

$$\|v_n - \bar{u}_n\|_X > c > 0 \tag{5.3}$$

where $\bar{u}_n := u_{n,\bar{\alpha}_n}$ and $\bar{\alpha}_n$ as before, for any n sufficiently large and for $\bar{\alpha} \neq \alpha_k$, recalling that $\bar{u}_n \to U_{\bar{\alpha}}$ in X by Proposition 3.2.
To prove (5.3) we argue by contradiction and assume that $v_n - \bar{u}_n \to 0$ in X. Then $\|v_n - \bar{u}_n\|_\infty \to 0$ as $n \to +\infty$, and this is not possible (see Proposition 4.4) since $\bar{\alpha} \neq \alpha_k$. Since we have reached a contradiction (5.3) holds. □

Remark 5.6. We remark that the bifurcation from the points (α_k, U_{α_k}) obtained in the Theorem 5.5 is indeed global. In fact we proved the existence of a closed connected set $\mathcal{C}_k$ that bifurcates from every point (α_k, U_{α_k}). Finally all solutions on the continuum $\mathcal{C}_k$ are fast decaying solutions of (1.1) because they decay as $\frac{1}{(1+|x|)^{N-2}}$ when $|x|$ is large, from Lemma 4.1. This indeed is a consequence of the fact that the solutions we find are the limit, in some sense, of the solutions of the approximating problem in a ball.

Proof of Theorem 1.6. Theorem 5.5 proves the existence of a continuum $\mathcal{C}_k$ of solutions of (1.1), invariant with respect to $O(N-1)$ bifurcating from (α_k, U_{α_k}) with $\alpha_k = 2(k-1)$ for any $k \geq 2$, and then proves i).
Moreover, when k is even, repeating the proof of Theorem 5.5 using the space

$$\mathcal{H}^h := \{v \in X \text{ s.t. } v \text{ is invariant by the action of } \mathcal{G}_h\}$$

for $h = 1, \dots, \left[\frac{N}{2}\right]$, with $\mathcal{G}_h$ as in (3.23), and using Remark 3.10 we find $\left[\frac{N}{2}\right]$ different continua bifurcating from (α_k, U_{α_k}). Each continuum is invariant with respect the action of $\mathcal{G}_h$ for some h and then ii) follows from (3.23).
Finally the decay of the solutions we find follows from Lemma 4.1 since the continua $\mathcal{C}_k$ are bounded by construction (see also Remark 5.6). □

5.2 An explicit solution

In this Section we construct an explicit branch of solutions to (1.1). The idea is the same as in Theorem 1.3.

We want to reduce our problem to another one where there is no dependence on $|x|^\alpha$ and the dimension $M = \frac{2(N+\alpha)}{2+\alpha}$, as we did in Section 2 (see (2.6)). Suppose that we have M integer and consider the known solutions when $\alpha = 0$ in $\mathbb{R}^M$,

$$U(x) = \frac{1}{(1+|x-y|^2)^{\frac{M-2}{2}}} = \frac{1}{(1+|x|^2 - 2x \cdot y + |y|^2)^{\frac{M-2}{2}}}$$

and setting $|x| = r$ we get

$$U(x) = \frac{1}{(1+r^2 - 2r|y|\cos \widehat{xy} + |y|^2)^{\frac{M-2}{2}}} \tag{5.4}$$

Proceeding as in Section 2, we consider the transformation $r \mapsto r^{\frac{2+\alpha}{2}}$ and $|y| = a$. However, in this case it is not clear how it transforms the angular term in (5.4). Thus we seek solutions of (1.1) in the form

$$u(x) = \frac{1}{(1+|x|^{2+\alpha} - 2aY(x) + a^2)^{\frac{M-2}{2}}} \tag{5.5}$$

for some function $Y(x)$. The proof of Theorem 1.3 suggests that the homogeneous harmonic polynomials of degree $k = \frac{2+\alpha}{2}$ can be good candidates, but (5.5) does not satisfy (1.1) for every choice of $Y \in \mathbb{Y}_k$. Substituting (5.5) into (1.1) one can check that $Y \in \mathbb{Y}_k$ must satisfy

$$|\nabla Y(x)|^2 = \left(\frac{2+\alpha}{2}\right)^2 |x|^\alpha. \tag{5.6}$$

Let us consider $\alpha = 2$ (the other cases are more involved), we have $k = 2$ and $M = 1 + \frac{N}{2}$ so we consider N even. Hence we want to find $Y \in \mathbb{Y}_2$ such that $|\nabla Y(x)|^2 = 4|x|^2$. In this case we readily see that $Y(x) = |x'|^2 - |x''|^2$ gives a solution, where we write $x \in \mathbb{R}^N$ as $x = (x', x'') \in \mathbb{R}^{\frac{N}{2}} \times \mathbb{R}^{\frac{N}{2}}$. Therefore the Proposition 1.7 is proved.

Note that the same result can be obtained applying the method used in [PS].

A Appendix. Some elementary proofs of known results

In this appendix we give a new proof of some known results. The first one is the following inequality (see E. Lieb [L83], B Gidas and J. Spruck [GS81])

Theorem A.1. *Let $u \in D^{1,2}\left(\mathbb{R}^N\right)$ be a radial function. Then we have that,*

$$\int_{\mathbb{R}^N} |\nabla u|^2 \geq C(\alpha, N) \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{\frac{2N+2\alpha}{N-2}}\right)^{\frac{N-2}{N+\alpha}} \tag{A.7}$$

Moreover the extremal functions which achieve $C(\alpha, N)$ are unique (up to dilations) and are given by

$$U_{\lambda,\alpha}(x) = \frac{\lambda^{\frac{N-2}{2}}}{\left(1 + \lambda^{2+\alpha} D |x|^{2+\alpha}\right)^{\frac{N-2}{2+\alpha}}} \tag{A.8}$$

with $\lambda > 0$ and a suitable $D \in \mathbb{R}^+$.

Proof. Let $u \in D^{1,2}\left(\mathbb{R}^N\right)$ be a radial function. Then we have that

$$\int_0^\infty \left|u'(r)\right|^2 r^{N-1} = \left(\text{setting } r = s^{\frac{2}{\alpha+2}} \text{ and } v(s) = u\left(s^{\frac{2}{\alpha+2}}\right)\right)$$

$$= \frac{\alpha+2}{2} \int_0^\infty \left|v'(s)\right|^2 s^{\frac{2N-2+\alpha}{\alpha+2}}.$$

Set

$$\frac{2N-2+\alpha}{\alpha+2} = M - 1 \tag{A.9}$$

which implies

$$\int_0^\infty |u'(r)|^2 r^{N-1} = \frac{\alpha+2}{2} \int_0^\infty |v'(s)|^2 s^{M-1}.$$

Now we use the classical Sobolev inequality (see Lemma 2 in [TA76]) and we get

$$\frac{\alpha+2}{2} \int_0^\infty |v'(s)|^2 s^{M-1} \geq \frac{\alpha+2}{2} S(M) \left(\int_0^\infty |v(s)|^{\frac{2M}{M-2}} s^{M-1} \right)^{\frac{M-2}{M}}$$

$$= \left(\frac{\alpha+2}{2}\right)^{\frac{2M-2}{M}} S(M) \left(\int_0^\infty |u(r)|^{\frac{2M}{M-2}} r^{\frac{M(\alpha+2)-2}{2}} \right)^{\frac{M-2}{M}}.$$

Here $S(M) = M(M-2) \left[\frac{\left(\Gamma(\frac{M}{2})\right)^2}{2\Gamma(M)} \right]^{\frac{2}{M}}$ (see [TA76]).

From (A.9) we deduce that $\frac{2M}{M-2} = \frac{2N+2\alpha}{N-2}$ and $\frac{M(\alpha+2)-2}{2} = N-1+\alpha$. So we get

$$\int_0^\infty |u'(r)|^2 r^{N-1} \geq \left(\frac{\alpha+2}{2}\right)^{\frac{2N-2+\alpha}{N+\alpha}} S\left(\frac{2N+2\alpha}{\alpha+2}\right) \left(\int_0^\infty r^\alpha |u(r)|^{\frac{2N+2\alpha}{N-2}} r^{N-1} \right)^{\frac{N-2}{N+\alpha}}$$

which proves (A.7) with

$$C(\alpha, N) = \left(\frac{\alpha+2}{2}\right)^{\frac{2N-2+\alpha}{N+\alpha}} S\left(\frac{2N+2\alpha}{\alpha+2}\right) \left(\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right)^{\frac{2+\alpha}{N+\alpha}}$$

Moreover, from the previous inequalities, we also get that the extremal functions are obtained as

$$\int_0^\infty |v'(s)|^2 s^{M-1} = S(M) \left(\int_0^\infty |v(s)|^{\frac{2M}{M-2}} s^{M-1} \right)^{\frac{M-2}{M}}$$

It is well-known that $v_\mu(s) = \frac{\mu^{\frac{M-2}{2}}}{(1+D^2\mu^2 s^2)^{\frac{M-2}{2}}}$ for some positive constant D and for any $\mu \in \mathbb{R}^+$. By (A.9) we get that the functions $U_{\lambda,\alpha}(x) = v\left(s^{\frac{\alpha+2}{2}}\right) = \frac{\mu^{\frac{N-2}{2+\alpha}}}{(1+D^2\mu^2|x|^{2+\alpha})^{\frac{N-2}{2+\alpha}}}$ are extremal for the inequality (A.7). Setting $\mu^2 = \lambda^{2+\alpha}$ we have (A.8). $\square$

Next result is a short proof of the existence and uniqueness result of the radial solution founded by W. M. Ni in [N82].

Theorem A.2. *Let B_1 the unit ball in $\mathbb{R}^N$ with $N \geq 3$. Then the problem*

$$\begin{cases} -\Delta u = |x|^\alpha u^p & \text{in } B_1 \\ u > 0 & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases} \tag{A.10}$$

admits a unique radial solution for $1 < p < \frac{N+2+2\alpha}{N-2}$ and any $\alpha \geq 0$.

Proof. Since u is radial we have that it satisfies,

$$\begin{cases} -u'' - \frac{N-1}{r} u' = r^\alpha u^p & \text{in } (0,1) \\ u > 0 & \text{in } (0,1) \\ u'(0) = u(1) = 0 \end{cases} \tag{A.11}$$

As in the previous theorems, let us consider the transformation

$$r = s^{\frac{2}{\alpha+2}} \quad \text{and} \quad v(s) = u\left(s^{\frac{2}{\alpha+2}}\right).$$

Then (A.11) becomes, setting $M = \frac{2(N+\alpha)}{2+\alpha}$

$$\begin{cases} -v''(s) - \frac{M-1}{r} v'(s) = \frac{4}{(2+\alpha)^2} v^p & \text{in } (0,1) \\ v > 0 & \text{in } (0,1) \\ v'(0) = v(1) = 0 \end{cases} \tag{A.12}$$

Note that we have $v'(0) = 0$ since $v'(s) = \frac{2}{\alpha+2} s^{-\frac{\alpha}{\alpha+2}} u'\left(s^{\frac{2}{\alpha+2}}\right)$ and by (A.11) $u'(r) = O(r^{\alpha+1})$ near $r = 0$. Moreover, since $1 < p < \frac{N+2+2\alpha}{N-2}$ we get $1 < p < \frac{M+2}{M-2}$.
Since the space $H = \left\{u \in H^1(0,1) \text{ such that } u(1) = 0\right\}$ equipped with the norm $||u||_H^2 = \int_0^1 |u'(s)|^2 s^{M-1} ds$ is compactly embedded in $L = \{u \in L^{p+1}(0,1)$ such that $\int_0^1 |u(s)|^{p+1} s^{M-1} ds\}$, for any $M > 0$ and $1 < p < \frac{M+2}{M-2}$, we have the existence of the solution.
The ODE (A.12) was considered in [GNN79], (Sec. 2.8, p. 224) when $1 < p < \frac{M+2}{M-2}$ and it was proved that it has a *unique* solution. In [GNN79] only the case $M \in \mathbb{N}$ was considered, but it is easy to see that the same proof applies in the general case $M \in \mathbb{R}$. So the same uniqueness result holds for problem (A.11) and the claim follows. □

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